Name: Solution Key Panther ID: ____________

FINAL EXAM Calculus I Spring 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (24 pts) Find the derivative of each of the following functions:

(a) $f(x) = 7x + 2^x - 3\pi^2$

$f'(x) = 7 + 2^x \cdot \ln 2$

(b) $f(x) = x^2 e^x$

$f'(x) = 2x e^x + x^2 e^x$

(c) $f(x) = \tan(\sec(5x))$

$f'(x) = \sec^2(\sec(5x)) \cdot \sec(5x) \cdot \tan(5x) \cdot 5$

(d) $f(x) = \sin^{-1} x - \sqrt{1+x^2}$ $= \arcsin x - (1+x^2)^{\frac{1}{2}}$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{x}{\sqrt{1+x^2}}$

2. (8 pts) Find $\frac{dy}{dx}$ where y is implicitly defined as a function of x by

$$x^3y^2 - 5x^2y + x = 1.$$

Take $\frac{d}{dx}$ of both sides.

$3x^2 \cdot y^2 + \underline{x^3 \cdot 2y \cdot y'} - 10x \cdot y - \underline{5x^2 \cdot y'} + 1 = 0$

$y'(2x^3y - 5x^2) = 10xy - 3x^2y^2 - 1$

$$\boxed{\frac{dy}{dx} = y' = \frac{10xy - 3x^2y^2 - 1}{x^2(2xy - 5)}}$$

$\lim_{x\to 0} \frac{\sin(3x)}{x} = 3$ or l'H again

3. (24 pts) Find the following limits, if they exist

(a) $\lim_{x\to 3} \frac{x^2-9}{|x-3|} = \lim_{x\to 3} \frac{(x-3)(x+3)}{|x-3|}$ $\frac{0}{0}$

consider one-sided limits (because of $|x-3|$)

$\lim_{x\to 3^+} \frac{(x-3)(x+3)}{|x-3|} = \lim_{x\to 3^+} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = 6$

$\lim_{x\to 3^-} \frac{(x-3)(x+3)}{|x-3|} = \lim_{x\to 3^-} \frac{\cancel{(x-3)}(x+3)}{-\cancel{(x-3)}} = -6$

Thus $\lim_{x\to 3} \frac{x^2-9}{|x-3|}$ D.N.E.

(b) $\lim_{x\to 0} \frac{1-\cos(3x)}{8x^2}$ $\frac{0}{0}$ $\overset{l'H}{=} \lim_{x\to 0} \frac{\sin(3x)\cdot 3}{16x} = \frac{9}{16}$

Sol. 2 (without l'H)

$\lim_{x\to 0} \frac{(1-\cos(3x))(1+\cos(3x))}{8x^2(1+\cos(3x))} = \lim_{x\to 0} \frac{\sin^2(3x)}{8x^2(1+\cos(3x))} =$

$= \frac{1}{8} \lim_{x\to 0}\left[\left(\frac{\sin(3x)}{x}\right)^2 \cdot \frac{1}{1+\cos(3x)}\right] = \frac{1}{8}\cdot 3^2 \cdot \frac{1}{2} = \frac{9}{16}$

(c) $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

Fastest sol: Observe the above limit is exactly the derivative definition of $f(x) = \sqrt{x}$

But $(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

Thus $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{2\sqrt{x}}$

If you did not see this you should multiply up & down by the conjugate, etc.

(d) $\lim_{x\to +\infty} x^{(1/\sqrt{x})}$ ∞^0

$\lim_{x\to +\infty} e^{\ln(x^{\frac{1}{\sqrt{x}}})} = e^{\lim_{x\to +\infty} \frac{\ln x}{\sqrt{x}}}$

$\lim_{x\to +\infty} \frac{\ln x}{\sqrt{x}} \overset{\frac{\infty}{\infty}}{\underset{l'H}{=}} \lim_{x\to +\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x\to +\infty} \frac{2\sqrt{x}}{x}$

$= \lim_{x\to +\infty} \frac{2}{\sqrt{x}} = 0$

Thus $\lim_{x\to +\infty} x^{(1/\sqrt{x})} = e^0 = \boxed{1}$

4. (18 pts) Find each indicated antiderivative:

(a) $\int (3 - \frac{1}{1+x^2} + 2\sqrt{x})\, dx = 3x - \arctan x + 2\cdot\frac{2}{3}x^{\frac{3}{2}} + c$

$= 3x - \arctan x + \frac{4}{3}x^{\frac{3}{2}} + c$

(b) $\int \frac{\cos(2x)}{3+\sin(2x)}\, dx = \int \frac{\frac{1}{2}dw}{w} =$

$w = 3 + \sin(2x)$

$dw = 2\cos(2x)dx$

$\frac{1}{2}dw = \cos(2x)dx$

$= \frac{1}{2}\ln|w| + c = \boxed{\frac{1}{2}\ln(3+\sin(2x)) + c}$

(c) $\int x^2 e^{-x^3}\, dx = \int e^w \cdot (-\frac{1}{3})dw =$

$w = -x^3$

$dw = -3x^2 dx$

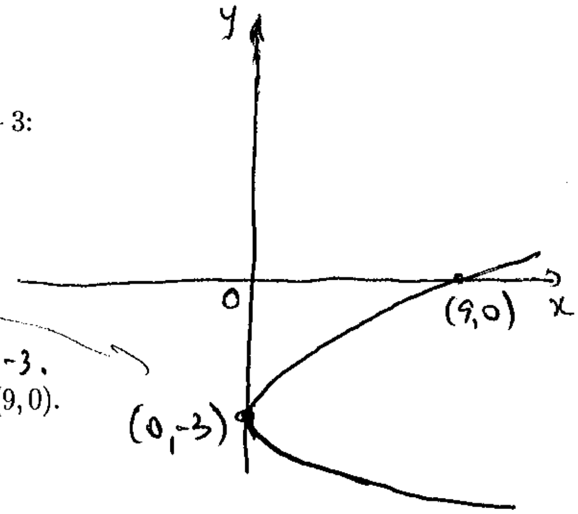
$-\frac{1}{3}dw = x^2 dx$

$= -\frac{1}{3}\int e^w dw = -\frac{1}{3}e^w + c =$

$\boxed{= -\frac{1}{3}e^{-x^3} + c}$

5. (12 pts) Given the parametric curve $x(t) = t^2$, $y(t) = t - 3$:

(a) (4 pts) Sketch the curve in the xy plane, clearly indicating orientation.



Eliminate parameter $t = y+3$

so $x = (y+3)^2$

parabola with horiz. axis of symmetry at $y=-3$.

(b) (8 pts) Find the tangent line to the curve at the point $(9, 0)$.

Point $(9,3)$ is obtained for $t=3$

$$m_{tan} = \left.\frac{dy}{dx}\right|_{t=3} = \frac{\left.\frac{dy}{dt}\right|_{t=3}}{\left.\frac{dx}{dt}\right|_{t=3}} = \frac{1}{2\cdot 3} = \frac{1}{6}$$

Tangent line: $y - 0 = \frac{1}{6}(x-9)$ $\boxed{y = \frac{1}{6}x - \frac{3}{2}}$

6. (8 pts) The sides of a cubic ice cube decrease at the rate of 0.2 cm/min. How fast is the surface area of the cube decreasing when the sides are 10 cm? Give units to your answer.

Let x = side of the cube

$x = x(t)$ $\quad \frac{dx}{dt} = -0.2$ cm/min

$S = 6x^2$ (as a cube has 6 square faces)

surface area

$$\frac{dS}{dt} = 12x\frac{dx}{dt} \Rightarrow \frac{dS}{dt} = 12\cdot 10(-.2) = -24 \frac{cm^2}{s}$$

Surface Area decreases at a rate of 24 $\frac{cm^2}{s}$.

7. (10 pts) Use limits to find all asymptotes (horizontal and vertical) of the function $f(x) = \frac{2x^2-x-1}{x^2-1}$. You are NOT required to graph this function.

Horiz. Asymptotes $\lim_{x\to+\infty} \frac{2x^2-x-1}{x^2-1} = 2$ & $\lim_{x\to-\infty} \frac{2x^2-x-1}{x^2-1} = 2$

Thus $\boxed{y=2}$ is a horiz. asymptote both when $x \to +\infty$ and $x \to -\infty$

V. Asymptotes $f(x) = \frac{(2x+1)(x-1)}{(x-1)(x+1)}$

$\lim_{x\to1} \frac{(2x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{3}{2}$ so $x=1$ is NOT a V. asymptote (it is a removable discontinuity)

$\lim_{x\to-1^-} \frac{(2x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{-1}{0^-} = +\infty$ so $\boxed{x=-1}$ is a V. asymptote

$\lim_{x\to-1^+} \frac{2x+1}{x+1} = \frac{-1}{0^+} = -\infty$

8. (12 pts) These are True or False questions. No partial credit. 2 points each.

a. A discontinuous function never has an absolute maximum. **True** **False**

b. If $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$, then f is continuous at $x = 2$. **True** **False**

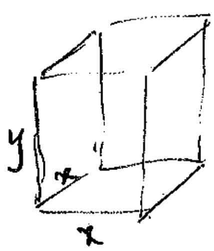
c. If $f'(2) = 0$ and $f''(2) < 0$ then f has a relative minimum at $x = 2$. **True** **False**

d. To compute the derivative of $\cos(\ln x)$ we must use the product rule. **True** **False**

e. If f is continuous at $x = 2$ then f is differentiable at $x = 2$. **True** **False**

f. If $\lim_{x\to 1} f(x) = 4$, then for x sufficiently close to 1, $f(x) < 4.01$. **True** **False**

9. (14 pts) A closed rectangular container with a square base is to have a volume of 2250 cubic inches. The material for the top and the bottom of the container will cost \$2 per square inch, and the material for the sides will cost \$3 per square inch. Find the dimensions of the container of least cost.



$$V = x^2 \cdot y = 2250 \Rightarrow y = \frac{2250}{x^2}$$

$$\underbrace{C}_{\text{Cost}} = 2\cdot(\underbrace{x^2+x^2}_{\text{area of top \& bottom}}) + 3\cdot(\underbrace{4xy}_{\text{area of the 4 sides}})$$

$$C = 4x^2 + 12xy$$

$$C(x) = 4x^2 + 12x\cdot\frac{2250}{x^2} = 4x^2 + \frac{12\cdot 2250}{x}$$

$$C'(x) = 8x - \frac{12\times 2250}{x^2}$$

$$C'(x) = 0 \iff 8x = \frac{12\times 2250}{x^2} \Rightarrow 8x^3 = 12\times 2250$$

$$\Rightarrow x^3 = \frac{12\times 2250}{8} = 3\times 1125 = 3\times 9\times 125$$

$$\Rightarrow x = \sqrt[3]{27\times 125} = 3\times 5 = 15$$

$$C''(x) = 8 + 2\cdot\frac{12\times 2250}{x^3} > 0$$ so $x = 15$ is indeed Abs. max,

$$y = \frac{2250}{15^2} = \frac{2250}{15\times 15} = 10$$

Optimal dimensions

$$\boxed{\begin{cases} x = 15 \\ y = 10 \end{cases}}$$

10. (16 pts) For $f(x) = x^4 - 6x^2 + 5$

(a) Find the intervals on which f is increasing; on which f is decreasing.

(b) Find the critical points and determine whether a relative minimum, relative maximum or neither occurs there.

(c) Find the intervals on which f is concave up; on which f is concave down.

(d) Find the coordinates of all inflection points.

(e) Graph the function.

(a)+(b) Domain of f - all reals; Note also that $f(x)$ is an <u>even</u> function

$f'(x) = 4x^3 - 12x = 4x(x^2-3) = 4x(x-\sqrt{3})(x+\sqrt{3})$

Critical points $x=0$, $x=\sqrt{3}$, $x=-\sqrt{3}$

x	$-\infty$		$-\sqrt{3}$	-1	0	1	$\sqrt{3}$		$+\infty$
$f'(x)$		$- - - - -$	0	$+ + +$	0	$- - -$	0	$+ + + + +$	
$f(x)$	$+\infty$	$\searrow$	-4	$\nearrow$	5	$\searrow$	-4	$\nearrow$	$+\infty$
$f''(x)$		$+ + + + + +$		0	$- - -$	0		$+ + + + +$	

So $x_1 = -\sqrt{3}$, $x_3 = \sqrt{3}$ are both rel. min.

$x_2 = 0$ is a rel. max.

f is decreasing on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

f is increasing on $(-\sqrt{3}, 0) \cup (\sqrt{3}, +\infty)$

(c)+(d)

$f''(x) = 12x^2 - 12 = 12(x-1)(x+1)$

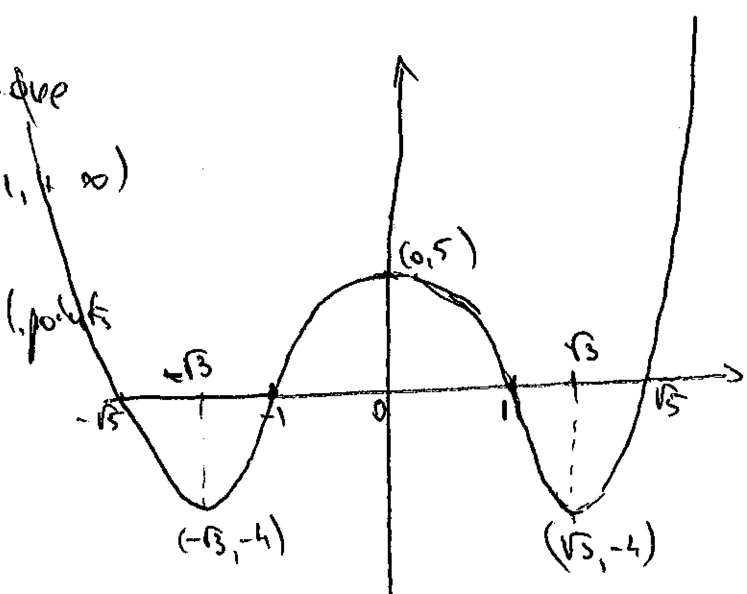
$f''(x) = 0 \iff x = 1$, $x = -1$

Sign chart for f'' above

$f(x)$ concave up on $(-\infty, -1) \cup (1, +\infty)$

$f(x)$ concave down on $(-1, 1)$

$x=-1$, $x=1$ are both infl. points



End behavior

$\lim_{x \to \pm\infty} (x^4 - 6x^2 + 5) = +\infty$

x-intercepts $0 = x^4 - 6x^2 + 5 = (x^2-5)(x^2-1)$

$x = \pm 1$, $x = \pm\sqrt{5}$

11. (10 pts) Use linear approximation (differentials) to approximate $\sqrt[3]{1003}$.

Approximate $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ at $x_0 = 1000$

$$f(x_0) = \sqrt[3]{1000} = 10$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3(\sqrt[3]{x})^2} \Rightarrow f'(x_0) = \frac{1}{3(\sqrt[3]{1000})^2} = \frac{1}{300}$$

Thus $f(x) = \sqrt[3]{x} \approx 10 + \frac{1}{300}(x - 1000)$

$$\sqrt[3]{1003} \approx 10 + \frac{1}{300}(1003 - 1000) = 10 + \frac{1}{100} = 10.01$$

12. (8 pts) At 11 A.M. on a certain day the outside temperature was 76° F. At 11 P.M. the temperature dropped to 52° F. Show that at some instant during this period the temperature was decreasing at the rate of 2° F/h. Specify the result you are using.

Let $T(t)$ be the temperature at time t (h). Let $t = 0$ be 11 A.M. Then $t = 12$ corresp. to 11 P.M.

As $T(t)$ is a continuous & differentiable function we can apply M.V.T.

Note that $T(0) = 76°$ $\quad T(12) = 52$

Thus the average rate of change of the temperature is $\frac{T(12) - T(0)}{12 - 0} = \frac{52 - 76}{12} = -2°F/h$

By M.V.T., there is a moment $t_0 \in (0, 12)$ so that $T'(t_0) = \frac{T(12) - T(0)}{12 - 0} = -2°F/h$

But $T'(t_0)$ represents the rate of change of temperature at the moment t_0.