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Spring Break Worksheet – due Wednesday, March 19 - MAC 2312, Spring 2014

1. (a) Use IBP to derive a reduction formula for

$$\int x^n e^{-wx}\,dx, \text{ where } n \text{ is a nonnegative integer and } w \text{ is a positive constant.}$$

Next, you will use your reduction formula in part (a) to show that

$$\int_0^{+\infty} x^n e^{-wx}\,dx = \frac{n!}{w^{n+1}}.$$

Follow these steps:

(b) Denote $I_n = \int_0^{+\infty} x^n e^{-wx}\,dx$. Compute directly $I_0 = \int_0^{+\infty} e^{-wx}\,dx$.

(c) Use l'Hopital to show that $\lim_{x\to+\infty} x^n e^{-wx} = 0$.

(d) Use the reduction formula from part (a) and the observation in (c), to get the recursive formula

$I_n = \frac{n}{w} I_{n-1}$, for all $n \geq 1$.

(e) From (d) and (b), conclude that $I_n = \frac{n!}{w^{n+1}}$.

Note: You have to trust me that the improper integral you computed is an important one. Hence, it was worth the effort!

(a) Parts $du = e^{-wx}dx \quad v = x^n$

$u = -\frac{1}{w}e^{-wx} \quad dv = nx^{n-1}dx$

Thus $\boxed{\int x^n e^{-wx}dx = -\frac{1}{w}x^n e^{-wx} + \frac{n}{w}\int x^{n-1}e^{-wx}dx}$

(b) $I_0 = \int_0^{+\infty} e^{-wx}dx = -\frac{1}{w}e^{-wx}\Big|_{x=0}^{x=+\infty} = \frac{1}{w}e^0 = \frac{1}{w}$ (since $\lim_{x\to+\infty} e^{-wx} = 0$ as $w>0$)

(c) $\lim_{x\to+\infty} x^n e^{-wx} = \lim_{x\to+\infty} \frac{x^n}{e^{wx}} \overset{\text{l'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to+\infty} \frac{nx^{n-1}}{we^{wx}} = \dots = 0$

(d) From (a) $\int_0^\infty x^n e^{-wx}dx = -\frac{1}{w}x^n e^{-wx}\Big|_{x=0}^{x=+\infty} + \frac{n}{w}\int_0^\infty x^{n-1}e^{-wx}dx$

Using also (c), we get $I_n = \frac{n}{w}I_{n-1}$, for $n \geq 1$.

(e) Use the relation from (d) repeatedly:

$I_n = \frac{n}{w}I_{n-1} = \frac{n}{w}\cdot\frac{(n-1)}{w}I_{n-2} = \dots = \frac{n(n-1)\cdot\ldots\cdot 1}{w^n} I_0 \overset{(b)}{=} \frac{n!}{w^n}\cdot\frac{1}{w} = \boxed{\frac{n!}{w^{n+1}}}$