Name: Solution Key Panther ID: ____________

Exam 1 Calculus II Spring 2014

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (12 pts) Circle the correct answer for (a) and (b). Fill in the answer for (c).

(a) The average value of the function $f(x) = 1/x$ on the interval $[1, 3]$ is

(i) $\frac{1}{2}$ (ii) $\frac{2}{3}$ (iii) $\frac{\ln 3}{2}$ (iv) $-\frac{1}{3}$ (v) none of the previous

(b) The expression

$$\frac{d}{dx}\left(\int_0^{x^2} e^{t^2}\,dt\right) \text{ is equivalent to}$$

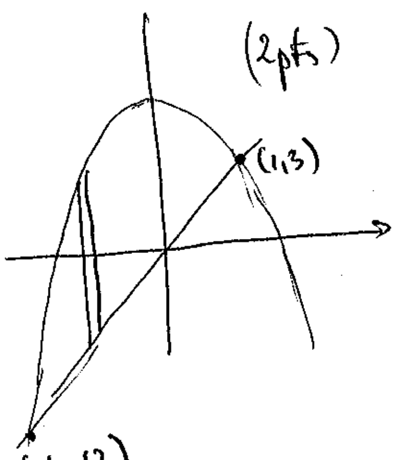
(i) e^{x^2} (ii) e^{t^2} (iii) $2xe^{x^4}$ (iv) e^{x^4} (v) $e^{x^4} - 1$

(c) A water-tank, initially full, starts being drained at the moment $t = 0$. Suppose that $r(t)$ (in gals/min) is the rate of the water flow out of the reservoir at the moment t.

In one sentence, explain what the equality $\int_0^{30} r(t)dt = 1000$ says in practical terms.

In the first 30 minutes, 1000 gals flowed out

2. (12 pts) Compute the area of the region bounded by the parabola $y = 4 - x^2$ and by the line $y = 3x$. Sketch of the region and full computation of the area are required for full credit.



Points of intersection

$$\begin{cases} y = 4 - x^2 \\ y = 3x \end{cases} \Rightarrow 4 - x^2 = 3x \Rightarrow$$

$$\Rightarrow x^2 + 3x - 4 = 0$$

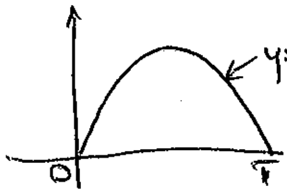
$$(x+4)(x-1) = 0 \Rightarrow \begin{cases} (x=-4,\ y=-12) \\ (x=1,\ y=3) \end{cases} \text{ (3pts)}$$

$$A = \int_{x=-4}^{x=1} (4 - x^2 - 3x)\,dx \quad \text{(3pts)}$$

$$A = \left(4x - \frac{x^3}{3} - \frac{3x^2}{2}\right)\Big|_{x=-4}^{x=1} \text{ (2pts)} = 20 - \left(\frac{1^3}{3} - \frac{(-4)^3}{3}\right) - \frac{3}{2}\left(1^2 - (-4)^2\right) \text{ (1pt)}$$

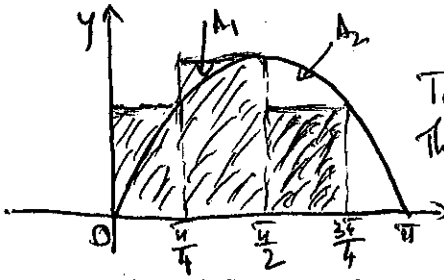
$$A = 20 - \frac{65}{3} + \frac{45}{2} = -\frac{5}{3} + \frac{45}{2} = \boxed{\frac{125}{6}} = \boxed{20\frac{5}{6}} \text{ (1pt)}$$

3. (10 pts) (a) (5 pts) Find the area below $y = \sin x$ and above the x-axis, for $x \in [0, \pi]$.



$$A = \int_0^{\pi} \sin x \, dx = -\cos x \Big|_{x=0}^{x=\pi} = -\cos\pi + \cos 0 = -(-1) + 1 = \boxed{2}$$

(b) (5 pts) On a graph of $y = \sin x$, for $x \in [0, \pi]$, draw and shade the approximation of the area in part (a) given by the right end-point Riemann sum with 4 equal subdivisions. Is it an over-estimate or an under-estimate of the area? Briefly justify.



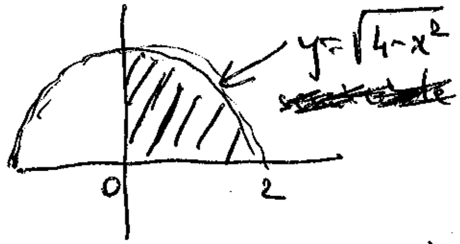
R_4 is, in this case, an <u>under-estimate</u> of the area in (

To see this, compare the intervals $[\frac{\pi}{4}, \frac{\pi}{2}]$ & $[\frac{\pi}{2}, \frac{3\pi}{4}]$. The area A_1 of the first "curved" triangle is less than the area A_2 of the second (because of the concavity of $y = \sin x$) Similarly, for intervals $[0, \frac{\pi}{4}]$ and $[\frac{3\pi}{4}, \pi]$.

4. (28 pts) Compute each integral (7 pts each):

(a) $\int_0^2 \sqrt{4 - x^2}\, dx$

Use geometry to evaluate this integral



$$\int_0^2 \sqrt{4-x^2}\, dx = \frac{1}{4}(\pi \cdot 2^2) = \boxed{\pi}$$

(b) $\int_0^1 \frac{1}{2x+1}\, dx$

$$\int \frac{1}{2x+1}\, dx = \frac{1}{2} \ln(2x+1)$$

? by guess & adjust . . .

or

$$\int_0^1 \frac{1}{2x+1}\, dx =$$

$u = 2x+1$

$du = 2dx$

$\frac{1}{2} du = dx$

$$= \int_{u=1}^{u=3} \frac{1}{u} \frac{1}{2} du = \frac{1}{2} \int_{u=1}^{u=3} \frac{1}{u}\, du$$

$$= \frac{1}{2} \ln|u| \Big|_{u=1}^{u=3} = \frac{1}{2}(\ln 3 - \ln 1)$$

$$= \boxed{\frac{1}{2} \ln 3}$$

(c) $\int_0^{\sqrt{\pi}/2} x\sec^2(x^2)\,dx$ $=$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \int_{u=0}^{u=\frac{\pi}{4}} \sec^2(u)\ \frac{1}{2}du$

$= \frac{1}{2}\tan u \Big|_{u=0}^{u=\frac{\pi}{4}}$

$= \frac{1}{2}\left(\tan\left(\frac{\pi}{4}\right) - \tan 0\right)$

$= \boxed{\frac{1}{2}}$

(d) $\int_{-4}^{0} x\sqrt{1-2x}\,dx$ $=$

$w = 1-2x \Rightarrow x = \frac{1-w}{2}$

$dw = -2\,dx$

$-\frac{1}{2}dw = dx$

$= \int_{w=9}^{w=1} \frac{1-w}{2}\cdot\sqrt{w}\cdot\left(-\frac{1}{2}\right)dw$

$= \frac{1}{4}\int_{w=1}^{w=9}(1-w)\,w^{\frac{1}{2}}\,dw$

$= \frac{1}{4}\int_1^9\left(w^{\frac{1}{2}} - w^{\frac{3}{2}}\right)dw$

$= \frac{1}{4}\left(\frac{2}{3}w^{\frac{3}{2}} - \frac{2}{5}w^{\frac{5}{2}}\right)\Big|_{w=1}^{w=9}$

$= \frac{1}{6}w^{\frac{3}{2}}\Big|_{w=1}^{w=9} - \frac{1}{10}w^{\frac{5}{2}}\Big|_{w=1}^{w=9}$

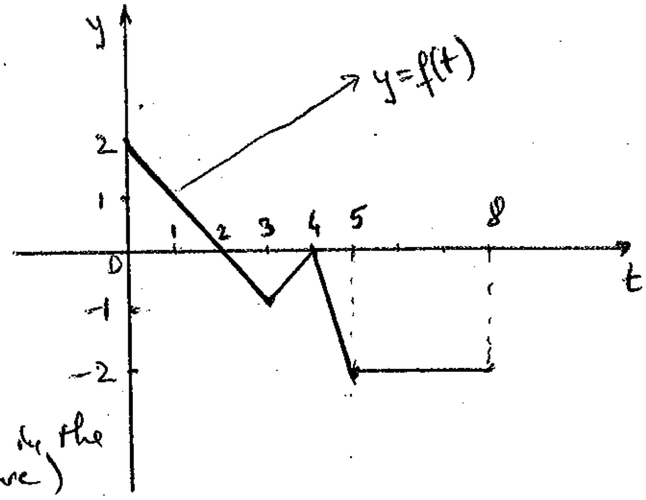
$3^5 = 243$

$= \frac{1}{6}\left(9^{\frac{3}{2}} - 1\right) - \frac{1}{10}\left(9^{\frac{5}{2}} - 1\right)$

$= \frac{26}{6} - \frac{242}{10} = \frac{13}{3} - \frac{121}{5}$

$= -\frac{298}{15} \approx -20 + \frac{2}{15}$

5. (10 pts) Let $F(x) = \int_0^x f(t)\,dt$, for $x \in [0,8]$ where f is the function whose graph is shown below.



(a) (6 pts) Find $F(0), F(3), F(5)$.

$F(0) = \int_0^0 f(t)dt = 0$

$F(3) = \int_0^3 f(t)dt = \int_0^2 f(t)dt + \int_2^3 f(t)dt$

$F(3) = \frac{2\cdot 2}{2} - \frac{1\cdot 1}{2} = 2 - \frac{1}{2} = 1{,}5$ (use areas in the picture)

$F(5) = \int_0^2 f(t)dt + \int_2^4 f(t)dt + \int_4^5 f(t)dt = \frac{2\cdot 2}{2} - \frac{2\cdot 1}{2} - \frac{2\cdot 1}{2} = \boxed{0}$

(b) (4 pts) Where does F have its absolute maximum value in the interval [0,8]? Its minimum value?

Since $f(t) = F'(t)$, the absolute max of F occurs at $t=2$.

The absolute min of F on $[0,8]$ occurs at $t=8$.

6. (10 pts) A stone dropped from the top of a building hits the ground with a speed of 96 ft/s. How tall is the building? Assume the initial velocity of the stone to be 0 and assume the gravitational acceleration $g = 32$ ft/s^2.

Use the equations $V = at + V_0$

$S = \frac{at^2}{2} + V_0t + S_0$

where $V_0 = 0$, $a = -g = -32\,\frac{ft}{s^2}$ and S_0 = initial height of the stone

so S_0 = height of the building.

From the impact velocity being $96\,\frac{ft}{s}$, we compute the time it takes the stone until it hits the ground.

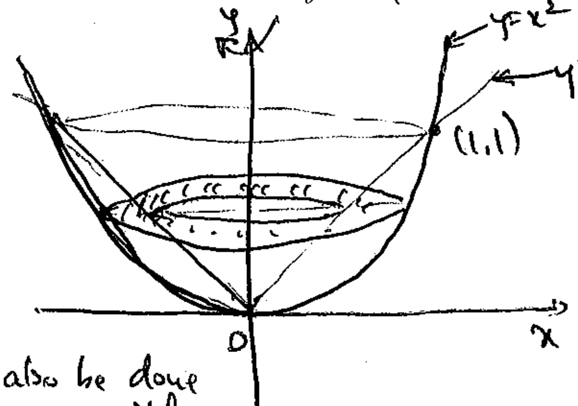
$-96 = -32\cdot t \Rightarrow t = \frac{-96}{-32} = \boxed{3\,s}$

We know that at $t = 3s$, $S = 0$. Thus

$0 = -\frac{32\cdot 3^2}{2} + S_0 \Rightarrow \boxed{S_0 = +16\cdot 9 = 144\,ft}$

7. (16 pts) Set up integrals to represent each of the following (you **do not** have to evaluate).

(a) (8 pts) The volume of the solid generated when the region in the first quadrant enclosed between $y = x$ and $y = x^2$ is revolved around the y-axis. (Sketch of solid is required.)



Intersection point $(1,1)$

with cross-section method

$$V = \int_?^? A_{slice} \cdot Th_{slice}$$

$$Th_{slice} = dy$$

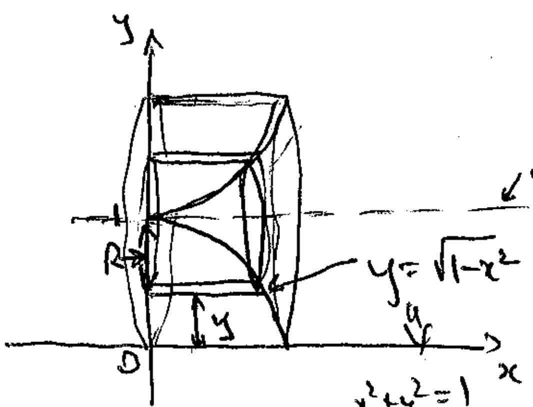
$$A_{slice} = \pi(R^2 - r^2) \quad \text{where } R = x_{parab.} = \sqrt{y}, \quad r = x_{line} = y$$

Note: Could also be done equally easy with cylindrical shells.

$$\text{so } V = \pi \int_{y=0}^{y=1} \left((\sqrt{y})^2 - (y)^2 \right) dy = \boxed{\pi \int_0^1 (y - y^2)\, dy}$$

(b) (8 pts) The volume of the solid when the region in the first quadrant enclosed by $y = \sqrt{1 - x^2}$ is rotated around the line $y = 1$. (Sketch of solid is required.)

I'll do this one with cylindrical shells, although it is equally easy to do it with the slicing method



$$V = \int_?^? 2\pi R_{shell} \cdot h_{shell} \cdot Th_{shell}$$

$$y = \sqrt{1-x^2} \Rightarrow x^2 + y^2 = 1 \Rightarrow x = \sqrt{1-y^2}$$

$$Th_{shell} = dy$$

$$R_{shell} = 1 - y$$

$$h_{shell} = x_{curve} = \sqrt{1 - y^2}$$

$$\boxed{V = 2\pi \int_0^1 (1-y)\sqrt{1-y^2}\, dy}$$