Name: Solution Key Panther ID: ____________

Exam 2 Calculus II Spring 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (a) (5 pts) Write the form of the partial fraction decomposition. Do **not** try to determine the constants.

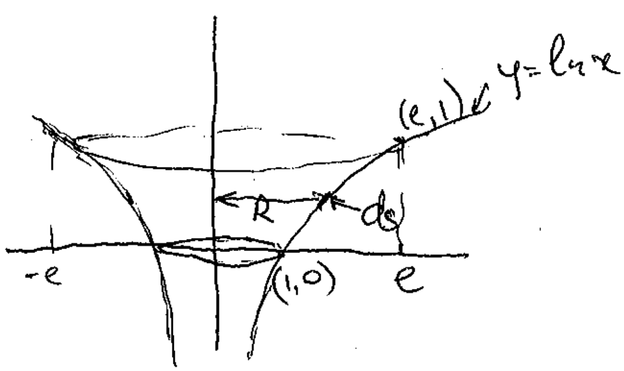
$$\frac{3x^5+1}{(x-1)^3(x^2+4)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

(b) (5 pts) Write the definitions of $\cosh x$, $\sinh x$ and state the basic identity that relates them.

$$\cosh x = \frac{e^x+e^{-x}}{2} \qquad \sinh x = \frac{e^x-e^{-x}}{2}$$

$$\cosh^2 x - \sinh^2 x = 1$$

2. (10 pts) Set up an integral that represents the area of the surface generated by revolving the graph of $y = \ln x$, $1 \leq x \leq e$, around the y-axis. Computation is **not** required, but a picture is.



$$S = \int_?^? 2\pi R\, ds$$

$$ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = 2\pi \int_{x=1}^{x=e} x\cdot\sqrt{1+\left(\frac{1}{x}\right)^2}\,dx$$

$$S = 2\pi \int_1^e x\cdot\sqrt{\frac{1+x^2}{x^2}}\,dx = 2\pi\int_1^e \sqrt{1+x^2}\,dx$$

Any one of these three forms is acceptable

3. (14 pts) Find the arc length of the curve $x = t^2/2\,,\;\; y = t^3/3$, for $t \in [0, 1]$. The computation is required.

4. (48 pts) Compute each of the following. (12 pts each):

(a) $\displaystyle\int_0^{\pi/6} \sin^3 x \, dx$

(b) $\displaystyle\int x \sec^2 x \, dx$

(c) $\displaystyle\int \frac{dx}{(a^2 - x^2)^{3/2}} \, dx$

(d) $\displaystyle\int \frac{x+1}{x^3 + x} \, dx$

5. (14 pts) A water tank has the shape of a circular cone with the flat side on the ground. The base has a diameter of 18 meters and the tip of the tank is 12 meters above the ground. Set up an integral that represents the work required to fill up the tank with water from ground level. The calculation is not required. Water has a density $\rho = 9810 \text{ N/m}^3$.

6. (14 pts) Choose ONE:

(a) State and prove the Work-Energy relationship.

See Text or notes.

(b) If n is a positive integer, derive a reduction formula for $\displaystyle\int_1^e (\ln x)^n \, dx$

$$\int_1^e 1 \cdot (\ln x)^n \, dx = *$$

use I.B.P.

$$du = 1 \cdot dx \qquad v = (\ln x)^n$$

$$u = x \qquad dv = n \cdot (\ln x)^{n-1} \cdot \frac{1}{x} dx$$

$$* = x (\ln x)^n \Big|_{x=1}^{x=e} - \int_{x=1}^{x=e} \cancel{x} \cdot n \cdot (\ln x)^{n-1} \cdot \frac{1}{\cancel{x}} dx =$$

$$= e - n \int_1^e (\ln x)^{n-1} dx$$

Thus

$$\boxed{\int_1^e (\ln x)^n dx = e - n \cdot \int_1^e (\ln x)^{n-1} dx}$$

Pb. 3.

$$s = \int_?^? ds \qquad ds = \sqrt{(dx)^2+(dy)^2}$$

in our case $x = x(t)$, $y = y(t)$ so

$$ds = \sqrt{(x'(t))^2+(y'(t))^2}\, dt$$

$$x'(t) = t \qquad y'(t) = t^2$$

So $$s = \int_{t=0}^{t=1} \sqrt{t^2+(t^2)^2}\, dt = \int_0^1 \sqrt{t^2+t^4}\, dt = \int_0^1 \sqrt{t^2(t^2+1)}\, dt$$

$$s = \int_0^1 t\sqrt{1+t^2}\, dt = \int_{w=1}^{w=2} w^{\frac{1}{2}} \tfrac{1}{2} dw = \tfrac{1}{\cancel{2}} \tfrac{\cancel{2}}{3} w^{\frac{3}{2}} \Big|_{w=1}^{w=2}$$

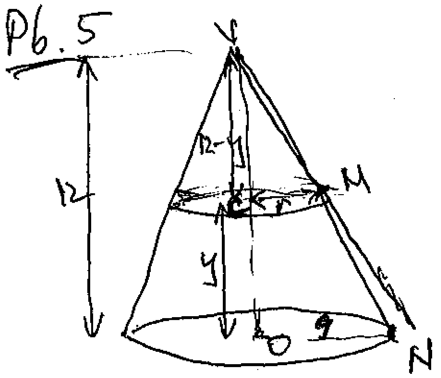
$w = 1+t^2$

$dw = 2t\,dt$

$\frac{1}{2} dw = t\,dt$

$$= \frac{1}{3}\left(2^{\frac{3}{2}}-1\right) = \boxed{\frac{1}{3}\left(\sqrt{8}-1\right)}$$

Pb. 5



$$W = \int_?^? dW$$

dW - work required to fill up one more "slice" of water of thickness dy above level y.

$$dW = \rho \cdot V_{slice} \cdot dist = \rho \cdot \pi r^2 \cdot th_{slice} \cdot dist$$

(ρ — density)

$th_{slice} = dy$

$dist = y$

r can be obtained from the similarity of ΔVCM with ΔVON

$$\frac{r}{9} = \frac{12-y}{12} \Rightarrow r = \frac{9}{12}(12-y) = \frac{3}{4}(12-y)$$

$$\boxed{W = \int_0^{12} \pi\rho \cdot \frac{9}{16}(12-y)^2 \cdot y\, dy}, \text{ where } \rho = 9810$$

Pb. 4

(a) $\int_0^{\frac{\pi}{6}} \sin^3 x \, dx = \int_0^{\frac{\pi}{6}} (1-\cos^2 x)\sin x \, dx =$

$w = \cos x$

$dw = -\sin x \, dx$

$$= -\int_1^{\frac{\sqrt{3}}{2}} (1-w^2)\, dw = \int_{\frac{\sqrt{3}}{2}}^{1} (1-w^2)\, dw =$$

$$= -\left(w - \frac{1}{3}w^3\right)\Big|_{w=1}^{w=\frac{\sqrt{3}}{2}} = -w\left(1-\frac{1}{3}w^2\right)\Big|_{w=1}^{w=\frac{\sqrt{3}}{2}}$$

$$= -\frac{\sqrt{3}}{2}\left(1-\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{4}\right) + 1\cdot\left(1-\frac{1}{3}\cdot 1\right) = -\frac{\sqrt{3}}{2}\cdot\frac{3}{4} + \frac{2}{3} = \boxed{-\frac{3\sqrt{3}}{8} + \frac{2}{3}}$$

4(b) $\int x \sec^2 x \, dx =$

I.B.P. $\quad du = \sec^2 x \, dx \qquad v = x$

$u = \int \sec^2 x \, dx = \tan x \qquad dv = 1 \cdot dx$

$$= x \tan x - \int \tan x \, dx$$

$$= \boxed{x \tan x - \ln|\sec x| + C}$$

If you did not memorize $\int \tan x \, dx$, you could do it as follows

$$\int \tan x = \int \frac{\sin x}{\cos x}\, dx = \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

$u = \cos x$

$du = -\sin x \, dx$

$$= \ln|(\cos x)^{-1}| + C =$$

$$= \ln|\sec x| + C$$

4(c) $\displaystyle\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = *$

$x = a\sin\theta$

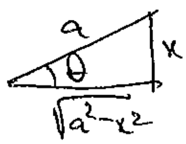
$(a^2-x^2)^{\frac{3}{2}} = (a^2 - a^2\sin^2\theta)^{\frac{3}{2}} = \left(a^2(1-\sin^2\theta)\right)^{\frac{3}{2}} = (a^2\cos^2\theta)^{\frac{3}{2}} = a^3\cos^3\theta$

$dx = a\cos\theta\, d\theta$

$$* = \int \frac{a\cos\theta\, d\theta}{a^3\cos^3\theta} = \frac{1}{a^2}\int \frac{1}{\cos^2\theta}\, d\theta = \frac{1}{a^2}\int \sec^2\theta\, d\theta$$

$$= \frac{1}{a^2}\tan\theta + C = \boxed{\frac{1}{a^2}\frac{x}{\sqrt{a^2-x^2}} + C}$$

$\sin\theta = \frac{x}{a} \Rightarrow \tan\theta = \frac{x}{\sqrt{a^2-x^2}}$



I use guessing and adjusting for finding the partial fraction decomposition below. More conventional methods will easily work too

4(d) $\displaystyle\frac{x+1}{x^3+x} = \frac{x+1}{x(x^2+1)} = \frac{\cancel{x}^{1}}{\cancel{x}(x^2+1)} + \frac{1}{x(x^2+1)} = \frac{1}{x^2+1} + \frac{1}{x} - \frac{x}{x^2+1}$

Thus $\displaystyle\int \frac{x+1}{x^3+x}\,dx = \int\left(\frac{1}{x^2+1} + \frac{1}{x} - \frac{x}{x^2+1}\right)dx$

$$\boxed{= \arctan x + \ln x - \frac{1}{2}\ln(x^2+1) + C}$$

Trig. sub. $x = \tan\theta$ is another approach for the problem