Name: Solution Key Panther ID: ____________

Exam 3 Calculus II Spring 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (10 pts) In each case answer True or False. No justification needed. (2 pts each)

(a) The alternating harmonic series is conditionally convergent.

True

(b) If $0 < a_k < \frac{1}{k}$ for all $k \geq 1$, then $\sum_{k=1}^{\infty} a_k$ is convergent.

False (Exp: $0 < a_k = \frac{1}{2k} < \frac{1}{k}$ but $\sum \frac{1}{2k} = \frac{1}{2}\sum \frac{1}{k} = +\infty$)

(c) The sequence $a_n = \sqrt{n} - 1000$, $n \geq 1$ is eventually positive.

True $a_n > 0$, when $n > 10^6$

(d) $1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \ldots = 1$

True. If S_n is the sum of the first n-terms

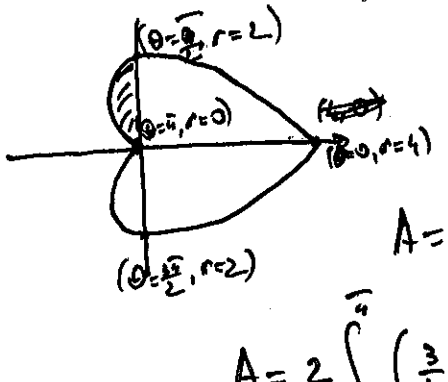
$$S_n = \begin{cases} 1 & \text{if } n = 2k+1 \\ 1 + \frac{1}{k} & \text{if } n = 2k \end{cases} \Rightarrow \lim_{n\to\infty} S_n = 1$$

(e) $1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \ldots = 1$

False

$$S_n = \begin{cases} 1 & \text{if } n = 2k+1 \\ 1 + \frac{1}{2} & \text{if } n = 2k \end{cases} \Rightarrow \lim_{n\to\infty} S_n \text{ D.N.E.}$$

2. (12 pts) Sketch the graph of the cardioid $r = 2 + 2\cos\theta$ by giving the coordinates of at least 4 points and then find the area bounded by the cardioid in the second quadrant.



$$A = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \left(2(1+\cos\theta)\right)^2 d\theta$$

$$A = 2\int_{\frac{\pi}{2}}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta = 2\int_{\frac{\pi}{2}}^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos(2\theta)}{2}\right) d\theta$$

$$A = 2\int_{\frac{\pi}{2}}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{\cos(2\theta)}{2}\right) d\theta = \int_{\frac{\pi}{2}}^{\pi} \left(3 + 4\cos\theta + \cos(2\theta)\right) d\theta$$

$$A = \frac{3\pi}{2} + 4\sin\theta \Big|_{\theta=\frac{\pi}{2}}^{\theta=\pi} + \frac{1}{2}\sin(2\theta)\Big|_{\theta=\frac{\pi}{2}}^{\theta=\pi} = \boxed{\frac{3\pi}{2} - 4}$$

3. (20 pts) Evaluate (or show it diverges) (5 pts each)

(*a*) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

$$\int \frac{1}{1+x^2} dx = \arctan x + c \quad \text{so} \quad \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \left(\lim_{x\to+\infty} \arctan x\right) - \left(\lim_{x\to-\infty} \arctan x\right)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$$

(*b*) $\lim_{k\to+\infty} \left(\frac{2}{3^k} - \frac{3}{2^k}\right) = 0$ since $3^k \to +\infty$ and $2^k \to +\infty$ when $k \to +\infty$

(*c*) $\sum_{k=0}^{+\infty} \left(\frac{2}{3^k} - \frac{3}{2^k}\right)$

$$\sum_{k=0}^{\infty} \frac{2}{3^k} = 2\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k \underset{\text{geometric}}{=} 2 \cdot \frac{1}{1-\frac{1}{3}} = \cancel{2} \cdot \frac{3}{\cancel{2}} = 3$$

$$\sum_{k=0}^{\infty} \frac{3}{2^k} = 3\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 3 \cdot \frac{1}{1-\frac{1}{2}} = 6$$

Since both series converge $\sum_{k=0}^{\infty}\left(\frac{2}{3^k} - \frac{3}{2^k}\right)$ converges to

$$3 - 6 = \boxed{-3}$$

(*d*) $\ln(1/3) + \ln(3/5) + \ln(5/7) + \ln(7/9) + \ldots$

$$= \sum_{k=0}^{\infty} \ln\left(\frac{2k+1}{2k+3}\right)$$

Sequence of partial sums

$$S_n = \sum_{k=0}^{n} \ln\left(\frac{2k+1}{2k+3}\right) = \sum_{k=0}^{n} \left(\ln(2k+1) - \ln(2k+3)\right) \overset{\text{telescopic}}{=}$$

$$= \left(\ln 1 - \cancel{\ln 3}\right) + \left(\cancel{\ln 3} - \cancel{\ln 5}\right) + \ldots + \left(\cancel{\ln(2n+1)} - \ln(2n+3)\right)$$

$$S_n = \ln 1 - \ln(2n+3) = -\ln(2n+3)$$

So $\lim_{n\to+\infty} S_n = -\infty$, so the series diverges.

4. (24 pts) For each of the following series, determine if the series is divergent (D), conditionally convergent (CC), or absolutely convergent (AC). Answer **and carefully** justify your answer. Very little credit will be given just for a guess. Most credit is given for the quality of the justification. (8 pts each)

(a) $\frac{1}{3} - \frac{3}{5} + \frac{5}{7} - \frac{7}{9} + \frac{9}{11} - \frac{11}{13} + \ldots$ $= \sum_{k=0}^{\infty} (-1)^k \frac{2k+1}{2k+3}$

but $\lim_{k \to +\infty} \frac{2k+1}{2k+3} = 1$

so $\lim_{k \to +\infty} (-1)^k \frac{2k+1}{2k+3}$ D.N.E. so the series <u>diverges</u> by the divergence test

(b) $\sum_{k=0}^{\infty} (-1)^k \frac{(k!)^2}{(2k+1)!}$ Try Absolute Ratio Test

$$\rho = \lim_{k \to +\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k \to +\infty} \frac{((k+1)!)^2}{(2k+3)!} \cdot \frac{(2k+1)!}{(k!)^2} =$$

$$= \lim_{k \to +\infty} \frac{(k+1)^2}{(2k+3)(2k+2)} = \frac{1}{4}$$

Since $\rho = \frac{1}{4} < 1$, the series is absolutely convergent by the absolute ratio test

(c) $\sum_{k=2}^{\infty} (-1)^k \frac{1}{k \ln k}$

Consider $\sum_{k=2}^{\infty} \left| (-1)^k \frac{1}{k \ln k} \right| = \sum_{k=2}^{\infty} \frac{1}{k \ln k}$ series diverges by the integral test

$$\int_2^{+\infty} \frac{1}{x \ln x} dx = \ln(\ln x) \Big|_{x=2}^{+\infty} = +\infty$$

so the original series is <u>not</u> absolutely convergent

But $\sum_{k=2}^{\infty} (-1)^k \frac{1}{k \ln k}$ is convergent by A.S.T. $\left(a_k = \frac{1}{k \ln k} \xrightarrow{k \to \infty} 0 \right)$

So $\sum (-1)^k \frac{1}{k \ln k}$ is <u>conditionally convergent</u>

5. (12 pts) Find the Taylor series at $x_0 = 1$ of the function $f(x) = \frac{1}{x}$.
(Your final answer should use the summation notation.)

$f(x) = x^{-1}$ $\quad f'(x) = (-1)x^{-2}$ $\quad f''(x) = (-1)\cdot(-2)x^{-3}$ $\quad f^{(3)}(x) = (-1)\cdot(-2)\cdot(-3)x^{-4}$

$f^{(k)}(x) = (-1)\cdot(-2)\cdot\ldots\cdot(-k)\,x^{-k-1} = (-1)^k\cdot k!\cdot x^{-k-1}$

so $f^{(k)}(1) = (-1)^k\cdot k!$ for all $k \geq 0$

By definition $T_{f,x_0}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

so, the Taylor series in our case is $\sum_{k=0}^{\infty} \frac{(-1)^k\cdot \cancel{k!}}{\cancel{k!}}(x-1)^k$

Second way

$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ ← replace in both sides x by $1-x$ $\quad \frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{k=0}^{\infty}(1-x)^k = \sum_{k=0}^{\infty}(-1)^k(x-1)^k$

6. (14 pts) Find the interval of convergence (with endpoints) of the series $\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$.

Absolute Ratio Test

$\rho = \lim_{k\to+\infty} \frac{|x|^{2k+3}}{2k+3}\cdot\frac{2k+1}{|x|^{2k+1}} = \lim_{k\to+\infty}\left(|x|^2\cdot\frac{2k+1}{2k+3}\right) = |x|^2$

If $\rho = |x|^2 < 1$ series is abs. convergent

$|x|^2 < 1 \Rightarrow |x| < 1 \Rightarrow x \in (-1, 1)$

Endpoints: $x = 1$ $\quad \sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}$ convergent by AST

$x = -1$ $\quad \sum_{k=0}^{\infty}(-1)^k\cdot\frac{(-1)^{2k+1}}{2k+1} = -\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}$ convergent by AST

Thus, the interval of convergence for the series is

$I = [-1, 1]$.

7. (12 pts) Choose ONE to prove:

(a) State and prove the divergence test for series. See text or notes

(b) State and prove the convergence part of the simple comparison test (that is, the part saying that if a series converges then the other series converges).

Simple Comparison Test: If $0 \leq a_k \leq b_k$, for all k and $\sum_{k=1}^{\infty} b_k$ is convergent then $\sum_{k=1}^{\infty} a_k$ is convergent.

Proof: Let $S = \sum_{k=1}^{\infty} b_k = \lim_{n\to\infty} \left(\sum_{k=1}^{n} b_k \right)$. S is a finite real number by the assumption that $\sum b_k$ converge

We have $0 \leq a_1 + a_2 + \dots + a_n \leq b_1 + b_2 + \dots + b_n \leq S$

where in the first and third inequality we use that $a_k \geq 0$, $b_k \geq 0$ and for the second we used that $a_k \leq b_k$.

Thus, if $S_n = a_1 + a_2 + \dots + a_n$, the above inequality implies that S_n is bounded.

But $S_n = a_1 + \dots + a_n \underset{\substack{\uparrow \\ a_{n+1} \geq 0}}{\leq} a_1 + \dots + a_n + a_{n+1} = S_{n+1}$, so S_n is also monotone. Thus $\{S_n\}_n$ is convergent, so $\sum_{k=1}^{\infty} a_k$ converges.

8. (10 pts) Consider the sequence defined recursively by $a_0 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$, for $n \geq 0$. Show that a_n is convergent and find its limit.

By induction: $0 < a_n < 2$ for all $n \geq 0$.

True for $n = 0$. $0 < a_0 = \sqrt{2} < 2$ ✓

Assume true for a_n; prove for a_{n+1}

$0 < a_{n+1} = \sqrt{2 + a_n} \underset{\substack{\uparrow \\ a_n < 2}}{<} \sqrt{2+2} = 2$

By induction: $a_n < a_{n+1}$

For $n = 0$, $a_0 = \sqrt{2} < a_1 = \sqrt{2 + \sqrt{2}}$ True

Assume true $a_n < a_{n+1}$; prove that $a_{n+1} < a_{n+2}$

But $a_{n+1} = \sqrt{2 + a_n} \underset{\substack{\uparrow \\ \text{since } a_n < a_{n+1} \\ \text{by assumption}}}{<} \sqrt{2 + a_{n+1}} = a_{n+2}$

Since $\{a_n\}_n$ is monotone & bounded it must be convergent.

Let $\ell = \lim_{n\to\infty} a_n$.

Taking limit $n \to \infty$ in $a_{n+1} = \sqrt{2 + a_n}$ get $\ell = \sqrt{2 + \ell} \Rightarrow \ell^2 = 2 + \ell \Rightarrow$

$\Rightarrow \ell^2 - \ell - 2 = 0 \Rightarrow (\ell - 2)(\ell + 1) = 0$

$\Rightarrow \ell = 2$ or $\ell = -1$ cannot have a negative limit. Thus $\boxed{\lim a_n = 2}$