Name: Solution Key Panther ID: ____________

FINAL EXAM Calculus II Fall 2013

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (16 pts) Evaluate

(a) $\int_0^2 x^2\sqrt{x^3+1}\,dx$ $= \int_{w=1}^{w=9} w^{\frac{1}{2}}\frac{1}{3}\,dw = \frac{1}{3}\int_1^9 w^{\frac{1}{2}}\,dw$

$w = x^3+1$

$dw = 3x^2\,dx$

$\frac{1}{3}dw = x^2\,dx$

$= \frac{1}{3}\frac{2}{3}w^{\frac{3}{2}}\Big|_{w=1}^{w=9}$

$= \frac{2}{9}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{2}{9}(26) = \frac{52}{9}$

(b) $\int_1^{+\infty} \frac{1}{(2x-1)^3}\,dx$ $= \lim_{b\to+\infty}\int_1^b \frac{1}{(2x-1)^3}\,dx =$

$\int \frac{1}{(2x-1)^3}\,dx = \int (2x-1)^{-3}\,dx = \frac{1}{(-2)}\cdot\frac{1}{2}(2x-1)^{-2} = -\frac{1}{4(2x-1)^2}$

Guess & adjust

(or do sub $w = 2x-1$)

$= \lim_{b\to+\infty}\left(\left(-\frac{1}{4(2x-1)^2}\right)\Big|_{x=1}^{x=b}\right) = \lim_{b\to+\infty}\left(-\frac{1}{4(2b-1)^2} + \frac{1}{4}\right) = \boxed{\frac{1}{4}}$

2. (18 pts) Circle the correct answer (work is not required for this problem) :

(a) Let $s(t)$ be the position of a particle in rectilinear motion during the time interval $a \le t \le b$. The total distance traveled by the particle is given by

(i) $\dfrac{s(b)-s(a)}{b-a}$ (ii) $s(b)$ (iii) $\displaystyle\int_a^b |s'(t)|\,dt$ (iv) $\displaystyle\int_a^b s(t)\,dt$ (v) $s(b)-s(a)$

$s'(t) = v(t)$ so $|s'(t)| = |v(t)| =$ spee

(b) For $x \in [0,1]$, the expression

$$\frac{d}{dx}\Big(\int_0^{x^2} \sqrt{1-t^2}\,dt\Big) \text{ is equivalent to}$$

(FTC + Chain Rule)

(i) $\sqrt{1-x^2}$ (ii) $2x-2x^3$ (iii) $2x\sqrt{1-x^4}$ (iv) $\sqrt{1-x^4}$ (v) $1-x^2$

(c) The average value of the function $f(x) = x^2$ over the interval $[0,2]$ is

(i) 2 (ii) $\frac{8}{3}$ (iii) 1 (iv) $\frac{4}{3}$ (v) 0

$$\text{Avg} = \frac{\int_0^2 x^2\,dx}{2} = \ldots$$

(d) Let $f(x)$ be a *linear* function and let T_4 be the trapezoid approximation with 4 subdivisions of the integral $\int_{-2}^{2} f(x)\,dx$. Then compared with the integral, T_4 is an

(i) overestimate (ii) underestimate (iii) exact estimate (iv) cannot tell (more should be known about f)

(e) For the integral $\int \sqrt{x^2-a^2}\,dx$, the following substitution is helpful:

(i) $x = a\sin\theta$ (ii) $w = x^2 - a^2$ (iii) $x = a\sec\theta$ (iv) $x = a\tan\theta$ (v) $w = (x-a)^2$

(f) The sequence $a_n = 2 + \dfrac{(-1)^n}{n}$, $n \ge 1$ is

(i) convergent but not monotone (ii) monotone but divergent (iii) bounded but divergent
(iv) eventually decreasing but unbounded (v) none of the above

3. (20 pts) Evaluate

(a) $\int xe^{-x}\,dx$ $\quad = -xe^{-x} + \int e^{-x}\,dx = \boxed{-xe^{-x} - e^{-x} + C}$

IBP

$du = e^{-x}\,dx \qquad v = x$

$u = \int e^{-x}\,dx = -e^{-x} \qquad dv = dx$

(b) $\int \frac{1}{x(x^2+1)}\,dx$ $\quad = \int \left(\frac{1}{x} + \frac{-x}{x^2+1}\right) dx =$

Partial fractions

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + x(Bx+C)$$

$$1 = (A+B)x^2 + C\cdot x + A$$

$$\Rightarrow \begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=0 \end{cases}$$

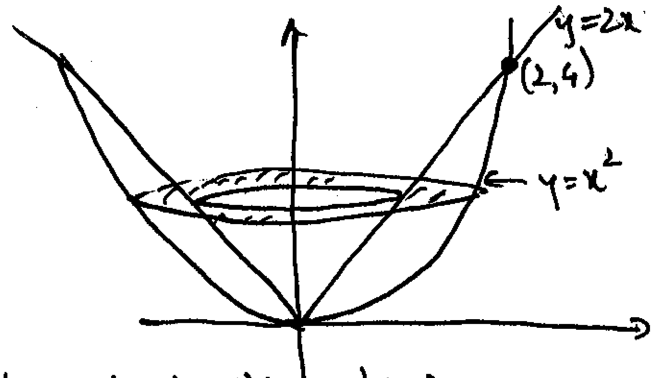
$= \ln x - \frac{1}{2}\ln(x^2+1) + C$

with guess & adjust for 2nd integral

may use sub. $w = x^2+1$ for that as well

4. (20 pts) In each case, sketch a picture and then set up an integral that represent each of the following. Computation of the integral is **not** required for this problem.

(a) Volume of the solid obtained when the region bounded by $y = 2x$ and $y = x^2$ is rotated around the y-axis.



with cross-section method

$$V = \int_?^? A_{slice} \cdot Th$$

$$Th = dy$$

$$A_{slice} = \pi (R^2 - r^2)$$

Inters. pt. $\begin{cases} y = 2x \\ y = x^2 \end{cases} \to \begin{cases} x = 2 \\ y = 4 \end{cases}$

$$R = x_{parab} = \sqrt{y}$$

$$r = x_{line} = \frac{y}{2}$$

$$\boxed{V = \pi \int_{y=0}^{y=4} \left(y - \frac{y^2}{4}\right) dy}$$

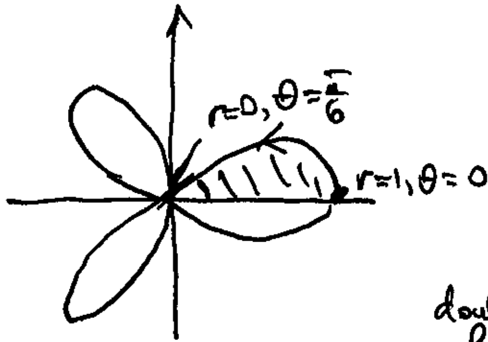
Cylindrical shells method works equally well.

(b) Arc-length of the parametric curve $x = t - 1, y = t^3$, when $-1 \le t \le 1$.

$$s = \int_?^? ds = \int_{t=-1}^{t=1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$s = \int_{-1}^{1} \sqrt{1 + (3t^2)^2}\, dt = \boxed{\int_{-1}^{1} \sqrt{1 + 9t^4}\, dt}$$

5. (12 pts) Find the area enclosed by one petal of the rose $r = \cos(3\theta)$. Full computation required.



$$A_{\frac{1}{2}\text{ petal}} = \frac{1}{2}\int_{\theta=0}^{\theta=\frac{\pi}{6}} \cos^2(3\theta)\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{1+\cos(6\theta)}{2}\,d\theta =$$

double angle formula

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} (1+\cos(6\theta))\,d\theta = \frac{1}{4}\left(\theta + \frac{1}{6}\sin(6\theta)\right)\Big|_{\theta=0}^{\theta=\frac{\pi}{6}} = \frac{1}{4}\cdot\frac{\pi}{6} + 0 = \frac{\pi}{24}$$

Thus $A_{\text{petal}} = 2\cdot\frac{\pi}{24} = \frac{\pi}{12}$

6. Choose ONE: (a)(15 pts) State and prove the second part of FTC (about $\frac{d}{dx}\int$);
(b) (10 pts) State and prove the integration by parts formula.

see text or notes

7. (26 pts) Determine whether each of the following series converges conditionally, converges absolutely, or diverges. Be sure to state which test you are using and to show that it applies to the series in question.

(a)(12 pts) $\frac{1}{1} - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \frac{5}{9} - \ldots$ Hint: First use summation notation.

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{2k-1}$$

Since $\lim_{k\to+\infty} \frac{k}{2k-1} = \frac{1}{2} \Rightarrow$

$\Rightarrow \lim_{k\to+\infty} (-1)^{k+1} \cdot \frac{k}{2k-1}$ D.N.E. so

$$\lim_{k\to+\infty} (-1)^{k+1} \frac{k}{2k-1} \neq 0 \underset{\text{div. test}}{\Longrightarrow} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{2k-1} \text{ diverges}$$

(b)(14 pts) $\sum_{k=2}^{+\infty} (-1)^{k+1} \frac{k}{k^2-1}$

Test Abs. convergence $\sum_{k=2}^{\infty} \left| (-1)^{k+1} \frac{k}{k^2-1} \right| = \sum_{k=2}^{\infty} \frac{k}{k^2-1}$ comparable with $\sum_{k=2}^{\infty} \frac{1}{k}$

since $\sum_{k=2}^{\infty} \frac{1}{k} = +\infty$ and since $\frac{k}{k^2-1} > \frac{1}{k}$, we have $\sum_{k=1}^{\infty} \frac{k}{k^2-1} = +\infty$

(by simple comparison)

Thus the original series is <u>not</u> abs. convergent

For the convergence of the original series apply A.S.T.

$a_k = \frac{k}{k^2-1}$ is decreasing (show a bit of work here)

and $\lim_{k\to+\infty} a_k = 0$

Thus $\sum_{k=2}^{\infty} (-1)^{k+1} \frac{k}{k^2-1}$ is convergent by A.S.T.

Thus the series is conditionally convergent

8. (22 pts) (a) (14 pts) Find the interval of convergence of the series

$\sum_{k=0}^{\infty} \frac{3^k}{k+1}(x-1)^{k+1}$. Be sure to check if the endpoints belong to the interval of convergence.

(b) (8 pts) Determine a function $f(x)$ whose Taylor series at $x_0 = 1$ is the series in part (a). Hint: Find first $f'(x)$.

(a) Apply Abs. Ratio Test

$$\rho = \lim_{k\to+\infty} \frac{\frac{3^{k+1}}{k+2}|x-1|^{k+2}}{\frac{3^k}{k+1}\cdot|x-1|^{k+1}} = \lim_{k\to+\infty}\left(3|x-1|\cdot\frac{k+1}{k+2}\right) = 3|x-1|$$

$$\rho<1 \iff 3|x-1|<1 \iff |x-1|<\frac{1}{3} \iff 1-\frac{1}{3}<x<1+\frac{1}{3}$$

So $\frac{2}{3}<x<\frac{4}{3}$

End points:

$x=\frac{2}{3}$ $\quad \sum_{k=0}^{\infty}\frac{3^k}{k+1}\left(-\frac{1}{3}\right)^{k+1} = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{3(k+1)}$ convergent by A.S.T.

$x=\frac{4}{3}$ $\quad \sum_{k=0}^{\infty}\frac{3^k}{k+1}\cdot\frac{1}{3^{k+1}} = \sum_{k=0}^{\infty}\frac{1}{3(k+1)}$ divergent (harmonic)

Thus $I = \left[\frac{2}{3},\frac{4}{3}\right)$

(b) Let $f(x) = \sum_{k=0}^{\infty}\frac{3^k}{k+1}(x-1)^{k+1}$

Then $f'(x) = \sum_{k=0}^{\infty}\frac{3^k}{\cancel{(k+1)}}\cancel{(k+1)}(x-1)^k = \sum_{k=0}^{\infty}(3x-3)^k \underset{\uparrow \text{ geom.}}{=} \frac{1}{1-(3x-3)} = \frac{1}{4-3x}$

Thus $\boxed{f(x) = \int\frac{1}{4-3x}\,dx = -\frac{1}{3}\ln(4-3x)}$

9. (16 pts) (a) (8 pts) Approximate $\frac{1}{\sqrt[3]{e}} = e^{-1/3}$, using the MacLaurin polynomial of degree 4 of $f(x) = e^x$.

(b) (8 pts) How small is the error in your approximation in part (a)? Recall $|R_n(x)| \le \frac{M}{(n+1)!}|x - x_0|^{n+1}$.

(a) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ $\qquad T_4 = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

Thus $e^{-\frac{1}{3}} \approx 1 - \frac{\frac{1}{3}}{1!} + \frac{\left(\frac{1}{3}\right)^2}{2!} - \frac{\left(\frac{1}{3}\right)^3}{3!} + \frac{\left(\frac{1}{3}\right)^4}{4!}$

(b) Since $f^{(k)}(x) = e^x$

on the interval $[-\frac{1}{3}, 0]$ the maximum value

of e^x is $e^0 = 1$. Thus $M = 1$

Thus Error $= |R_4(x)| \le \frac{1}{(4+1)!} \left|-\frac{1}{3} - 0\right|^5 = \frac{\left(\frac{1}{3}\right)^5}{5!} = \frac{1}{3^5 \cdot 5!}$

~~Error~~