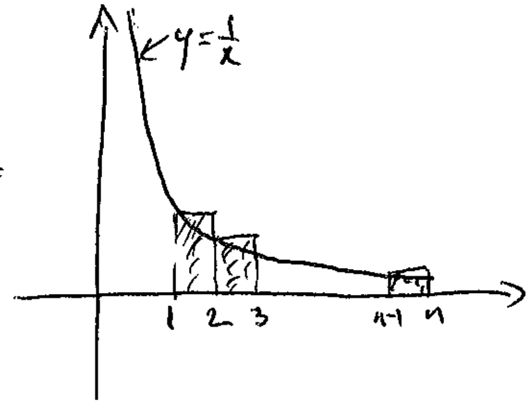
Worksheet week 11 MAC-2312 – Spring 2014 **Name:** Solution Key

1. Among other things, in this exercise you'll prove $\sum_{k=1}^{\infty} 1/k = +\infty$.

(a) Let $n \geq 2$ be an integer.
Find the area below the graph of $f(x) = 1/x$ and above the x-axis on the interval $[1, n]$.
(your answer will, of course, depend on n).

$$A = \int_1^n \frac{1}{x}\,dx = (\ln x)\Big|_{x=1}^{x=n} = \ln n - \ln 1 = \ln n$$



(b) Shade the left endpoint Riemann sum approximation of the area in part (a) for the division of the interval $[1, n]$ into sub-intervals of length 1.
Write the expression for this Riemann sum.

$$\text{L.R.S.} = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n-1}$$

(c) Explain why from parts (a) and (b) you obtain the inequality

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} \geq \ln n\,, \text{ and further, from this, } \sum_{k=1}^{+\infty} \frac{1}{k} = +\infty\,.$$

The L.R.S. is an over-estimate for $A = \int_1^n \frac{1}{x}dx$

so $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} \geq \ln n$

Thus $\sum_{k=1}^{n-1} \frac{1}{k} \geq \ln n$ Taking $n \to +\infty$

we get $\sum_{k=1}^{\infty} \frac{1}{k} = \lim_{n \to +\infty} \left(\sum_{k=1}^{n-1} \frac{1}{k} \right) \geq \lim_{n \to +\infty} \ln n = +\infty$

In the remaining parts of this exercise you'll show that the sequence

$$a_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} - \ln n \text{ is convergent when } n \to \infty.$$

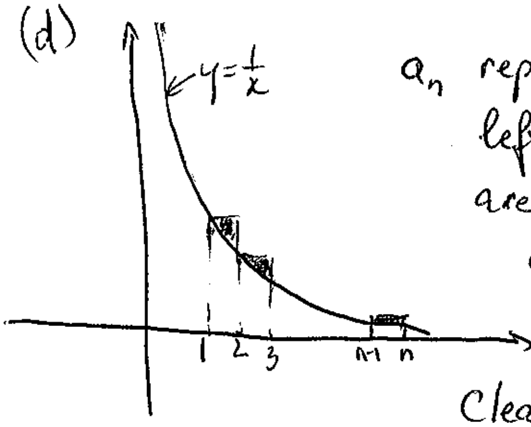
(The limit is the so called Euler's γ constant, $\gamma \approx 0.5772\ldots$.)

(d) Relating to parts (a) and (b) of the exercise, determine what a_n represents geometrically. If you understand this, you should be able to see that $a_n > 0$, for all $n \geq 2$ and that the sequence $\{a_n\}$ is strictly increasing. Briefly explain why these statements are true.

(e) Use the right-hand Riemann sum to show that $a_n < 1$, for all $n \geq 2$.

(f) From parts (c) and (d) it follows that the sequence $\{a_n\}$ is convergent. In one sentence explain why.

(d)



a_n represents the difference between the left end-point Riemann sum and the area below the curve $\frac{1}{x}$ on the interval $[1,n]$

a_n is the sum of the areas of the little "curved" triangles in the picture

Clearly, $a_n > 0$ (since the L.R.S. is an over-estimate)

Also, $a_{n+1} > a_n$, as one more little triangle will be added in the interval $[n, n+1]$.

(e) The right end-point R. sum is $\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ and is an underestimate of the area. Thus

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n} < \ln n \Rightarrow$$

$$\Rightarrow \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n} < 1 + \ln n \Rightarrow$$

$$\Rightarrow \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} - \ln n < 1 - \frac{1}{n} < 1$$

Thus $a_n < 1$ for any n.

(f) By (d), (e), $\{a_n\}_n$ is monotone and bounded, so it must be convergent.

2. For each of the following series, first write them using summation notation, then find a closed form for the n-th term S_n of the sequence of partial sums and then finally determine whether the series converges:

(a) $\ln\frac{1}{2}+\ln\frac{2}{3}+\ln\frac{3}{4}+\ln\frac{4}{5}+\ ...$ (b) $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}+\ ...$

Note: Both series are what we call "telescopic series".

(a) $$\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right) = \lim_{n\to+\infty} S_n \text{, where}$$

$$S_n = \sum_{k=1}^{n} \ln\left(\frac{k}{k+1}\right) = \sum_{k=1}^{n} (\ln k - \ln(k+1))$$

$$S_n = (\ln 1 - \cancel{\ln 2}) + (\cancel{\ln 2} - \cancel{\ln 3}) + (\cancel{\ln 3} - \cancel{\ln 4}) + \cdots + (\cancel{\ln n} - \ln(n+1))$$

$$S_n = \ln 1 - \ln(n+1) = -\ln(n+1)$$

Thus $$\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right) = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} (-\ln(n+1)) = -\infty$$

so series diverges.

(b) $$\sum_{k=2}^{+\infty} \frac{1}{k^2-1} = \sum_{k=2}^{\infty} \frac{1}{(k-1)(k+1)}$$

but $$\frac{1}{(k-1)(k+1)} = \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$$ (Guess & Adjust)

so $$S_n = \sum_{k=2}^{n} \frac{1}{(k-1)(k+1)} = \sum_{k=2}^{n} \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$$

$$S_n = \frac{1}{2}\left[\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{n-2}} - \frac{1}{n}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right.\right.$$

$$S_n = \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right]$$

$$\sum_{k=2}^{\infty} \frac{1}{k^2-1} = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right] = \boxed{\frac{3}{4}}$$