Name: Solution Key Panther ID: ____________

Midterm Exam MAT 3501 Fall 2016

1. (30 ~~25~~ pts) For each of the following statements answer if it is True or False. Then give a **one line** justification of your answer. (2 pts answer, 3 pts justification)

(a) The only consecutive integers that are both prime are 2 and 3. **True** **False**
(Recall that 1, by definition, is not prime nor composite.)

Justification: In ~~the~~ a pair $n, n+1$ one of them is even so, if $n>2$, not prime

(b) $(x-2)$ is a factor of $p(x) = x^4 - 6x - 4$. **True** **False**

Justification: $p(2) = 2^4 - 6\cdot 2 - 4 = 0$ so $x-2$ is a factor of $p(x)$ by Factor Thm.

(c) For all a, b rational numbers, a^b is rational. **True** **False**

Justification: $2^{\frac{1}{2}} = \sqrt{2}$ is irrational (or many other examples)

(d) For all integers l, m, n, if $l|(mn)$ then $l|m$ or $l|n$. **True** **False**

Justification: $6 | (3\cdot 4)$, but $6 \nmid 3$ and $6 \nmid 4$.

(e) If a, b are integers and $gcd(a, b) = 1$, then $lcm(a, b) = ab$. **True** **False**

Justification: $gcd(a,b) \cdot lcm(a,b) = ab$

(f) If p is prime and $p \geq 5$, then $(p+1)\,|\,p!$. **True** **False**

Justification: $p+1 = 2k$ with $3 \leq k < p$, ~~but~~ and $p! = 1\cdot 2 \cdot \ldots \cdot k \cdot \ldots \cdot p = (2k)\cdot l$ with $l \in \mathbb{Z}$.

2. (10 pts) Find **all** roots of the equation $x^3 + 2x + 3 = 0$. (Hint: the equation has a rational root.)

By rational root thm, ~~the rational~~ we could test if ± 1 or ± 3 are roots.

Since $(-1)^3 + 2\cdot(-1) + 3 = 0$, we see that $x_1 = -1$ is a root.

Then $(x+1)\overline{\big)\,x^3 + 2x + 3}$ with quotient $x^2 - x + 3$:

$$-x^3 - x^2$$
$$-x^2 + 2x + 3$$
$$+x^2 + x$$
$$3x + 3$$
$$-3x - 3$$
$$0$$

so $x^3 + 2x + 3 = (x+1)(x^2 - x + 3)$

By quadratic formula the other roots are

$$x_{2,3} = \frac{1 \pm \sqrt{1-12}}{2} = \frac{1 \pm i\sqrt{11}}{2}$$

Thus $\boxed{x_1 = -1,\quad x_2 = \frac{1 - i\sqrt{11}}{2},\quad x_3 = \frac{1 + i\sqrt{11}}{2}}$

3. (10 pts) Let $x_1, x_2, x_3 \in \mathbf{C}$ be the roots of the polynomial $p(x) = 2x^3 + 3x + 1$. Use Viete's relations to find:

(a) $x_1 + x_2 + x_3$

(b) $x_1 x_2 x_3$

(c) $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$

Viéte's relations for $ax^3 + bx^2 + cx + d = 0$

$$x_1 + x_2 + x_3 = -\frac{b}{a}$$

$$x_1x_2 + x_2x_3 + x_3x_1 = \frac{c}{a}$$

$$x_1x_2x_3 = -\frac{d}{a}$$

So, for our polynomial,

(a) $x_1 + x_2 + x_3 = 0$

(b) $x_1x_2x_3 = -\frac{1}{2}$

(c) $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{x_2x_3 + x_3x_1 + x_1x_2}{x_1x_2x_3} = \frac{\frac{3}{2}}{-\frac{1}{2}} = -3$

4. (16 pts) The product rule in Calculus states that $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

(a) (6 pts) Show that there is a product rule for 3 functions and that it is

$$(f(x)g(x)h(x))' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$\big(f(x)g(x)h(x)\big)' = \big((f(x)g(x))\cdot h(x)\big)' \underset{\substack{\uparrow \\ \text{usual prod. rule for}}}{=} (f(x)g(x))'h(x) + (f(x)g(x))\cdot h'(x)$$

$$\underset{\substack{\uparrow \\ \text{one more time prod. rule}}}{=} (f'(x)g(x) + f(x)g'(x))h(x) + f(x)g(x)h'(x) =$$

$$= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

(b) (10 pts) Generalize the product rule for n functions and prove it by induction.

Generalized product rule:

If $n \geq 2$ and $f_1(x), f_2(x), \dots, f_n(x)$ are n functions

$$(f_1(x)f_2(x)\dots f_n(x))' = f_1'(x)f_2(x)\dots f_n(x) + f_1(x)f_2'(x)\dots f_n(x) + \dots + f_1(x)f_2(x)\dots f_{n-1}(x)f_n'(x)$$

Proof: By induction on n.

Basis Case: $n=2$ $\to$ is the usual product rule

Inductive Step: Assume the rule holds for products of n functions and we'll prove it for products of $(n+1)$ functions

$$\big(f_1(x)f_2(x)\dots f_n(x)f_{n+1}(x)\big)' = \big[(f_1(x)f_2(x)\dots f_n(x))\cdot f_{n+1}(x)\big]' =$$

$$\underset{\substack{\uparrow \\ \text{usual prod. rule}}}{=} (f_1(x)\dots f_n(x))'f_{n+1}(x) + (f_1(x)\dots f_n(x))\cdot f_{n+1}'(x) =$$

$$\underset{\substack{\uparrow \\ \text{inductive assumption}}}{=} \big[f_1'(x)f_2(x)\dots f_n(x) + f_1(x)f_2'(x)\dots f_n(x) + \dots + f_1(x)f_2(x)\dots f_n'(x)\big]f_{n+1}(x) + f_1(x)\dots f_n(x)f_{n+1}'(x)$$

$$= f_1'(x)f_2(x)\dots f_n(x)f_{n+1}(x) + f_1(x)f_2'(x)\dots f_n(x)f_{n+1}(x) + \dots + f_1(x)f_2(x)\dots f_n(x)f_{n+1}'(x)$$

5. (10 pts) Show that, for any positive integer n, the greatest common divisor of $2n+13$ and $n+7$ is 1. As a consequence, justify that the fraction $\frac{n+7}{2n+13}$ is always in lowest terms.

~~Suppose~~ Let $d = \gcd(2n+13, n+7)$

Then $d \mid (n+7)$ & $d \mid (2n+13)$ $\Rightarrow$ $d \mid 2\cdot(n+7)$, $d \mid (2n+13)$ $\Rightarrow$ $\left.\begin{matrix} d \mid (2n+14) \\ d \mid (2n+13) \end{matrix}\right\} \Rightarrow$

$\Rightarrow d \mid (2n+14)-(2n+13)$ so $d \mid 1$. Thus $\underline{d=1}$

As $n+7$ and $2n+13$ have no common factor (other than 1)

$\frac{n+7}{2n+13}$ is in lowest terms for any $n>0$.

6. (10 pts) Using mods, find the remainder of $2016^{2015} + 2015^{2016}$ when divided by 7.

$2016 = 7 \times 288$ so $2016 \equiv 0 \pmod 7$

thus $2016^{2015} \equiv 0 \pmod 7$

Also $2016 \equiv 0 \pmod 7 \Rightarrow 2015 \equiv -1 \pmod 7$

so $2015^{2016} \equiv (-1)^{2016} \equiv 1 \pmod 7$

Thus $2016^{2015} + 2015^{2016} \equiv 1 \pmod 7$

so, when divided by 7, the number $2016^{2015} + 2015^{2016}$ gives ~~has a~~ the remainder $\underline{1}$.

7. (24 pts) Choose TWO of the following three (12 pts each)

(A) State and prove the Rational Root Theorem (it's OK if you give the detailed proof for just 1/2 of it).

(B) State and prove the quadratic formula.

(C) Prove that there are infinitely many primes (you can assume the prime factorization theorem).

See notes or textbook