6. The Diophantine equation $x^4 + y^4 + z^4 = 2673$ can be solved with positive integral values in two different ways; find them.

7. Find the number of different ways that the following numbers can be written as the sum of two cubes.

   (a) 4104
   (b) 46683

8. From *Arithmetica* by Diophantos. Express these questions as Diophantine systems and try to find solutions.

   (a) Find two square numbers such that when one forms their product and adds either of the numbers to it, the result is a square.
   (b) Find three numbers in an arithmetic series such that the sum of any two of them is a perfect square.
   (c) Find three numbers such that their sum is a square and the sum of any two of them is a square.

9. Find four solutions to the Diophantine equation $x^2 + y^2 = z^3$. Do you see a pattern you can extend?

10. Find five primitive solutions to each of these equations. A solution x, y, z is primitive if $\gcd(x, y, z) = 1$.

   (a) $x^2 + 2y^2 = z^2$
   (b) $x^2 + y^2 = 2z^2$

11. Examine consecutive squares in the following vein: $10^2 + 11^2 + 12^2 = 13^2 + 14^2$ is an example of five consecutive squares where the sum of the first three equals the sum of the last two.

   (a) Find an example of three consecutive squares where the sum of the first two equals the third.
   (b) Find an example of seven consecutive squares where the sum of the first four equals the sum of the final three.
   (c) Using the examples, look for a pattern and try to extend it.

12. Find the $\gcd(a, b)$ and then solve the linear Diophantine equation $ax + by = \gcd(a, b)$ where

   (a) $a = 28644,\ b = 13566$
   (b) $a = 57171,\ b = 116109$
   (c) $a = 101556,\ b = 605682$

13. In each of the following equations, find the smallest x and y such that

   (a) $889x + 511y = 63$
   (b) $5720x + 6171y = 77$
   (c) $9702x + 2873y = 103$

14. A friend of mine owns several 10-ounce silver pieces, each worth \$57. I own several 1-ounce gold pieces each worth \$375.

   (a) If I owed her money, what is the smallest amount I could pay off with coin swapping? Describe how I would pay this amount.
   (b) If she owed me money, what is the smallest amount she could pay off? Describe how she would do it.

15. Two different types of weights are used to ascertain the weight of a 6-ounce rock on a balance scale. One type of weight is a 20-pound 5-ounce weight, the other is a 33-pound 11-ounce weight. Find the smallest number of these two types of weights that are necessary to do the weighing.

16. Two stars are unaccountably flaring up periodically. Star A flares at intervals of 44, hours 39 minutes, and 57 seconds. Star B flares at intervals of 72 hours, 27 minutes, and 45 seconds. Star A just flared and star B followed just a minute later. Is this the closest interval of successive flares? If not, find a closer interval and tell when it will occur.

17. Finish the following statements and then prove them: Assume that a, b, and c are integers.

   (a) The line $ax + by = c$ in the xy plane passes through the point (x_0, y_0), where x_0 and y_0 are integers, if and only if

$x = 4 + 6n$ for some n. This is a contradiction. Thus the system has no solution. ■

Some of the most famous mathematical puzzles have been problems that translate into solving linear congruences or systems of lin- ear congruences. In particular, the ancient Indian, Chinese, and Greek mathematicians have explored such problems. Here are two examples. Several more are included in the exercises.

Example 2.2.22

This puzzle is attributed to Mahaviracarya, an Indian mathe- natician (about 850 A.D.).

There are 63 equal piles of plantain fruit along with 7 single ruits. They are evenly divided among 23 travelers. What is the umber of fruits in each pile? Let x represent the number of fruits ach traveler gets. Let y be the number of fruits in each pile. The ollowing equation represents the situation.

$$\text{(i)} \quad 23x = 63y + 7$$

Ve may write this as a linear congruence $23x \equiv 7 \pmod{63}$. Let us everse this and write $63y \equiv -7 \pmod{23}$. This simplifies to $17y \equiv 16$ nod 23). We may substitute values in for $y, y = 0, 1, 2, \ldots, 22$ ntil we solve this or we may use the method of reduction of moduli. ewriting this yields

$$\text{(ii)} \quad 17y = 16 + 23k \text{ for some integer } k.$$

gain using congruence terminology we get $23k \equiv -16 \pmod{17}$, hich simplifies to $6k \equiv 1 \pmod{17}$. At this point we may solve e congruence empirically; $k = 3$. Substituting back in equation), we have $17y = 16 + 23(3)$, so $y = 5$. Substituting this back into uation (i), we have $23x = 63(5) + 7$, so $x = 14$. This means that ch traveler gets 14 fruits and there are 5 fruits in each pile.

The general solution can be found this way: $k = 3 + 17l$. Substituting this back in (ii), we have $17y = 16 + 23(3 + 17l) = 85 + 23(17)l$. viding by 17, we get $y = 5 + 23l$. And substituting in (i) $23x = (5 + 23l) + 7$ yields $x = 14 + 63l$. If we let $l = 1$, we have 77 fruits each traveler, and 28 fruits in each pile. This is also ...

Here is another problem that has been around, in some form, for centuries. It supposedly originated in ancient China. It involves a system of linear congruences. We repeat it from the introduction.

Example 2.2.23

A band of 17 pirates stole a sack of gold coins. When they tried to divide the fortune into equal portions, 3 coins were left over. In the fight that followed one pirate was killed. The coins were redistributed into equal portions and this time 10 coins were left over. Another fight broke out and another pirate was killed. Now the coins could evenly be distributed. What is the number of coins that were stolen?

Letting x represent the number of coins, the congruences that describe this situation are $x \equiv 3 \pmod{17}$ $x \equiv 10 \pmod{16}$ $x \equiv 0 \pmod{15}$. This means

$$\begin{aligned} &\text{(i)} && x = 3 + 17k \\ &\text{(ii)} && x = 10 + 16j \\ &\text{(iii)} && x = 15h. \end{aligned}$$

Solving (i) and (ii) simultaneously, we get $17k = 7 + 16j$. Reducing modulo 16, we get $k \equiv 7 \pmod{16}$; or $k = 7 + 16l$. Now plugging this into (i), we get $x = 3 + 17k = 3 + 17(7 + 16l)$. So

$$\text{(iv)} \quad x = 122 + 272l.$$

Solving (iii) and (iv) simultaneously, we get $272l = -122 + 15h$. Reducing, modulo 15, we get $2l \equiv 13 \pmod{15}$; so $l \equiv 14 \pmod{15}$; that is, $l = 14 + 15m$. Thus $x = 122 + 272(14 + 15m)$. So

$$\text{(v)} \quad x = 3930 + 4080m.$$

Letting $m = 0$, we have the smallest number of coins: $x = 3930$. ■

Notice that in both of the preceding examples, 2.2.20 and 2.2.23, the difference between successive solutions to the linear congruences follows a formula. We have, in Example 2.2.20, the term $330t$ and $330 = 2 \times 3 \times 5 \times 11$, the product of the respective moduli. Also in Example 2.2.23 we have the term $4080m$ and $4080 = 15 \times 16 \times 17$, the product of the respective moduli. These example suggest a general theorem about the solution to linear congruences. It is called, appropriately, the Chinese remainder theorem after the many