Name: Solution key   Panther ID: ____________

Exam 1   MAT 3501   Fall 2018

1. (15 pts) For each of the following, answer if the statement is True or False. Then give a one line justification of your answer. (2 pts answer, 3 pts justification)

(a) The product of any two irrational numbers is irrational.   **True**   **False**

Justification: $\sqrt{2} \notin \mathbb{Q}$, but $\sqrt{2} \cdot \sqrt{2} = 2 \in \mathbb{Q}$

(b) If n is composite, then n has a divisor $d \neq 1$ so that $d \leq \sqrt{n}$.   **True**   **False**

Justification:

If $n = a \cdot b$ then either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ (otherwise, if $a > \sqrt{n}$ and $b > \sqrt{n}$, $a \cdot b = n > \sqrt{n}$ contradiction)

(c) In base 2, $\overline{111} + \overline{111} = \overline{1110}$.   **True**   **False**

Justification:

$\overline{111}_2 = 1 + 1 \cdot 2 + 1 \cdot 2^2$, so $\overline{111}_2 + \overline{111}_2 = 2(1 + 1 \cdot 2 + 1 \cdot 2^2) = 0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 = \overline{1110}_2$

2. (15 pts) Find the prime factorization for each of the numbers 216 and 2016.
Then find gcd(216, 2016), lcm(216, 2016).

$216 = 2^3 \cdot 3^3$

$2016 = 2^5 \cdot 3^2 \cdot 7$

$\gcd(216, 2016) = 2^3 \cdot 3^2 = 72$

$\operatorname{lcm}(216, 2016) = 2^5 \cdot 3^3 \cdot 7 = 6048$

3. (15 pts) Prove (by induction, or otherwise) that for all $n \geq 0$, $3^{2n+1} + 2^{n+2}$ is divisible by 7.

Here is a non-induction proof of the statement:

$$3^{2n+1} + 2^{n+2} = 3 \cdot 3^{2n} + 2^2 \cdot 2^n = 3 \cdot 9^n + 2^2 \cdot 2^n$$

But $9 \equiv 2 \pmod 7$ so

$$9^n \equiv 2^n \pmod 7 \quad \text{so}$$

$$3 \cdot 9^n + 2^2 \cdot 2^n \equiv 3 \cdot 2^n + 4 \cdot 2^n \equiv 7 \cdot 2^n \equiv 0 \pmod 7$$

Thus $3^{2n+1} + 2^{n+2} \equiv 0 \pmod 7$

so $3^{2n+1} + 2^{n+2}$ is divisible by 7.

4. (15 pts) (a) (10 pts) Show that for a number written in base 10, $\overline{a_n a_{n-1} ... a_2 a_1 a_0}$, we have

$$\overline{a_n a_{n-1} ... a_2 a_1 a_0} \equiv a_0 - a_1 + a_2 - ... + (-1)^n a_n \pmod{11} \ .$$

(b) (5 pts) Apply part (a) to find the remainder of 987654321 when divided by 11.

a)

$$\overline{a_n a_{n-1} \ldots a_2 a_1 a_0} = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \ldots + a_{n-1} \cdot 10^{n-1} + a_n \cdot 10^n$$

but $10 \equiv (-1) \pmod{11}$

so $10^2 \equiv (-1)^2 \equiv 1 \pmod{11}$

$10^3 \equiv (-1)^3 \equiv -1 \pmod{11}$

$- \quad -$

$10^n \equiv (-1)^n \pmod{11}$

Using these, we have

$$\overline{a_n a_{n-1} \ldots a_2 a_1 a_0} \equiv a_0 + a_1 \cdot (-1) + a_2 \cdot 1 + a_3 \cdot (-1) + \ldots + a_n (-1)^n \pmod{11}$$

(b) Using (a)

$$987654321 \equiv 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 \equiv 5 \pmod{11}$$

so the remainder when 987654321 is divided by 4 is $\underline{5}$.

5. (15 pts) Show that if the prime decomposition of a number is $N = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$, with $p_1, p_2, \ldots, p_k$ distinct primes, then the number of divisors of N (including 1 and N) is given by $(n_1+1)(n_2+1)\ldots(n_k+1)$.

A divisor d of N will have the form

$$d = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \quad \text{with } \underline{0 \le m_1 \le n_1, \ldots, 0 \le m_k \le n_k}. \ (*)$$

Thus, there is a one-to-one correspondence between divisors of N and the tuples $(m_1, m_2, \ldots, m_k)$ satisfying the condition $(*)$.

But there are n_1+1 possibilities for m_1 $(m_1 \in \{0,1,2,\ldots,n_1\})$, likewise there are n_2+1 possibilities for $m_2, \ldots, n_k+1$ possibilities for m_k.

Therefore there are $(n_1+1)(n_2+1)\cdots(n_k+1)$ possible tuples $(m_1, m_2, \ldots, m_k)$ satisfying conditions $(*)$, thus there are $(n_1+1)(n_2+1)\ldots(n_k+1)$ divisors of N.

6. (15 pts) Describe all integer solutions (if any) of the Diophantine equation $9x + 7y = 5$.

First note that $\gcd(7,9)=1$ and $1 \mid 5$ so the equation

$9x+7y=5$ does have integer solutions.

From the main theorem on Diophantine linear equations we know that if (x_0, y_0) is one solution (with $x_0, y_0 \in \mathbb{Z}$) then all solutions will be of the form $(x_\ell = x_0 - 7\ell,\ y_\ell = y_0 + 9\ell)$ with $\ell \in \mathbb{Z}$.

To find one solution one could use congruencies as we did in class.

$$9x+7y=5 \implies 9x \equiv 5 \pmod 7 \quad \text{or}$$

$$2x \equiv 5 \pmod 7$$

One solution for this is $x_0 = 6$ and plugging in the original equation, get $y_0 = -7$.

Thus, all solutions of the Diophantine equation are

$$(x_\ell = 6 - 7\ell,\ y_\ell = -7 + 9\ell) \text{ with } \ell \in \mathbb{Z}.$$

7. (15 pts) Choose ONE of the following proofs. If you do both, the second score may give some bonus towards a previous problem with a lower score.

(A) Show that if a, b are positive integers, then there exist integers m, n so that $ma + nb = \gcd(a, b)$.

(B) Show that if p is a prime number and $p|(ab)$ then $p|a$ or $p|b$. (You may use the result in (B) for proving (C).)

See your notes