components of X that are different from P and intersect C; each of them necessarily lies in C, so that

$$C = P \cup Q.$$

Because X is locally path connected, each path component of X is open in X. Therefore, P (which is a path component) and Q (which is a union of path components) are open in X, so they constitute a separation of C. This contradicts the fact that C is connected. ∎

Exercises

1. What are the components and path components of $\mathbb{R}_\ell$? What are the continuous maps $f : \mathbb{R} \to \mathbb{R}_\ell$?

2. (a) What are the components and path components of $\mathbb{R}^\omega$ (in the product topology)?
(b) Consider $\mathbb{R}^\omega$ in the uniform topology. Show that $\mathbf{x}$ and $\mathbf{y}$ lie in the same component of $\mathbb{R}^\omega$ if and only if the sequence

$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \ldots)$$

is bounded. [*Hint:* It suffices to consider the case where $\mathbf{y} = \mathbf{0}$.]
(c) Give $\mathbb{R}^\omega$ the box topology. Show that $\mathbf{x}$ and $\mathbf{y}$ lie in the same component of $\mathbb{R}^\omega$ if and only if the sequence $\mathbf{x} - \mathbf{y}$ is "eventually zero." [*Hint:* If $\mathbf{x} - \mathbf{y}$ is not eventually zero, show there is homeomorphism h of $\mathbb{R}^\omega$ with itself such that $h(\mathbf{x})$ is bounded and $h(\mathbf{y})$ is unbounded.]

3. Show that the ordered square is locally connected but not locally path connected. What are the path components of this space?

4. Let X be locally path connected. Show that every connected open set in X is path connected.

5. Let X denote the rational points of the interval $[0, 1] \times 0$ of $\mathbb{R}^2$. Let T denote the union of all line segments joining the point $p = 0 \times 1$ to points of X.
(a) Show that T is path connected, but is locally connected only at the point p.
(b) Find a subset of $\mathbb{R}^2$ that is path connected but is locally connected at none of its points.

6. A space X is said to be ***weakly locally connected at x*** if for every neighborhood U of x, there is a connected subspace of X contained in U that contains a neighborhood of x. Show that if X is weakly locally connected at each of its points, then X is locally connected. [*Hint:* Show that components of open sets are open.]

7. Consider the "infinite broom" X pictured in Figure 25.1. Show that X is not locally connected at p, but is weakly locally connected at p. [*Hint:* Any connected neighborhood of p must contain all the points a_i.]

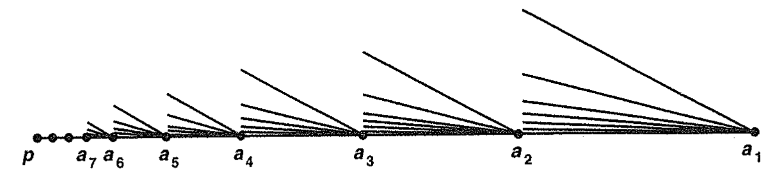

Figure 25.1

8. Let $p : X \to Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected. [*Hint:* If C is a component of the open set U of Y, show that $p^{-1}(C)$ is a union of components of $p^{-1}(U)$.]

9. Let G be a topological group; let C be the component of G containing the identity element e. Show that C is a normal subgroup of G. [*Hint:* If $x \in G$, then xC is the component of G containing x.]

10. Let X be a space. Let us define $x \sim y$ if there is no separation $X = A \cup B$ of X into disjoint open sets such that $x \in A$ and $y \in B$.
(a) Show this relation is an equivalence relation. The equivalence classes are called the ***quasicomponents*** of X.
(b) Show that each component of X lies in a quasicomponent of X, and that the components and quasicomponents of X are the same if X is locally connected.
(c) Let K denote the set $\{1/n \mid n \in \mathbb{Z}_+\}$ and let $-K$ denote the set $\{-1/n \mid n \in \mathbb{Z}_+\}$. Determine the components, path components, and quasicomponents of the following subspaces of $\mathbb{R}^2$:

$$A = (K \times [0, 1]) \cup \{0 \times 0\} \cup \{0 \times 1\}.$$
$$B = A \cup ([0, 1] \times \{0\}).$$
$$C = (K \times [0, 1]) \cup (-K \times [-1, 0]) \cup ([0, 1] \times -K) \cup ([-1, 0] \times K).$$

§26 Compact Spaces

The notion of compactness is not nearly so natural as that of connectedness. From the beginnings of topology, it was clear that the closed interval $[a, b]$ of the real line had a certain property that was crucial for proving such theorems as the maximum value theorem and the uniform continuity theorem. But for a long time, it was not clear how this property should be formulated for an arbitrary topological space. It used to be thought that the crucial property of $[a, b]$ was the fact that every infinite subset of $[a, b]$ has a limit point, and this property was the one dignified with the name of compactness. Later, mathematicians realized that this formulation does not lie at the heart of the matter, but rather that a stronger formulation, in terms of open coverings of the space, is more central. The latter formulation is what we now call compactness.

It is not as natural or intuitive as the former; some familiarity with it is needed before its usefulness becomes apparent.

Definition. A collection $\mathcal{A}$ of subsets of a space X is said to ***cover*** X, or to be a ***covering*** of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an ***open covering*** of X if its elements are open subsets of X.

Definition. A space X is said to be ***compact*** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

EXAMPLE 1. The real line $\mathbb{R}$ is not compact, for the covering of $\mathbb{R}$ by open intervals

$$\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$$

contains no finite subcollection that covers $\mathbb{R}$.

EXAMPLE 2. The following subspace of $\mathbb{R}$ is compact:

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}.$$

Given an open covering $\mathcal{A}$ of X, there is an element U of $\mathcal{A}$ containing 0. The set U contains all but finitely many of the points $1/n$; choose, for each point of X not in U, an element of $\mathcal{A}$ containing it. The collection consisting of these elements of $\mathcal{A}$, along with the element U, is a finite subcollection of $\mathcal{A}$ that covers X.

EXAMPLE 3. Any space X containing only finitely many points is necessarily compact, because in this case every open covering of X is finite.

EXAMPLE 4. The interval $(0, 1]$ is not compact; the open covering

$$\mathcal{A} = \{(1/n, 1] \mid n \in \mathbb{Z}_+\}$$

contains no finite subcollection covering $(0, 1]$. Nor is the interval $(0, 1)$ compact; the same argument applies. On the other hand, the interval $[0, 1]$ *is* compact; you are probably already familiar with this fact from analysis. In any case, we shall prove it shortly.

In general, it takes some effort to decide whether a given space is compact or not. First we shall prove some general theorems that show us how to construct new compact spaces out of existing ones. Then in the next section we shall show certain specific spaces are compact. These spaces include all closed intervals in the real line, and all closed and bounded subsets of $\mathbb{R}^n$.

Let us first prove some facts about subspaces. If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to ***cover*** Y if the union of its elements *contains* Y.

Lemma 26.1. *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.*

Proof. Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X. Then the collection

$$\{A_\alpha \cap Y \mid \alpha \in J\}$$

is a covering of Y by sets open in Y; hence a finite subcollection

$$\{A_{\alpha_1} \cap Y, \ldots, A_{\alpha_n} \cap Y\}$$

covers Y. Then $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ is a subcollection of $\mathcal{A}$ that covers Y.

Conversely, suppose the given condition holds; we wish to prove Y compact. Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y. For each α, choose a set A_α open in X such that

$$A'_\alpha = A_\alpha \cap Y.$$

The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X. By hypothesis, some finite subcollection $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ covers Y. Then $\{A'_{\alpha_1}, \ldots, A'_{\alpha_n}\}$ is a subcollection of $\mathcal{A}'$ that covers Y. ∎

Theorem 26.2. *Every closed subspace of a compact space is compact.*

Proof. Let Y be a closed subspace of the compact space X. Given a covering $\mathcal{A}$ of Y by sets open in X, let us form an open covering $\mathcal{B}$ of X by adjoining to $\mathcal{A}$ the single open set $X - Y$, that is,

$$\mathcal{B} = \mathcal{A} \cup \{X - Y\}.$$

Some finite subcollection of $\mathcal{B}$ covers X. If this subcollection contains the set $X - Y$, discard $X - Y$; otherwise, leave the subcollection alone. The resulting collection is a finite subcollection of $\mathcal{A}$ that covers Y. ∎

Theorem 26.3. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subspace of the Hausdorff space X. We shall prove that $X - Y$ is open, so that Y is closed.

Let x_0 be a point of $X - Y$. We show there is a neighborhood of x_0 that is disjoint from Y. For each point y of Y, let us choose disjoint neighborhoods U_y and V_y of the points x_0 and y, respectively (using the Hausdorff condition). The collection $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X; therefore, finitely many of them $V_{y_1}, \ldots, V_{y_n}$ cover Y. The open set

$$V = V_{y_1} \cup \cdots \cup V_{y_n}$$

contains Y, and it is disjoint from the open set

$$U = U_{y_1} \cap \cdots \cap U_{y_n}$$

formed by taking the intersection of the corresponding neighborhoods of x_0. For if z is a point of V, then $z \in V_{y_i}$ for some i, hence $z \notin U_{y_i}$ and so $z \notin U$. See Figure 26.1.

Then U is a neighborhood of x_0 disjoint from Y, as desired. ∎

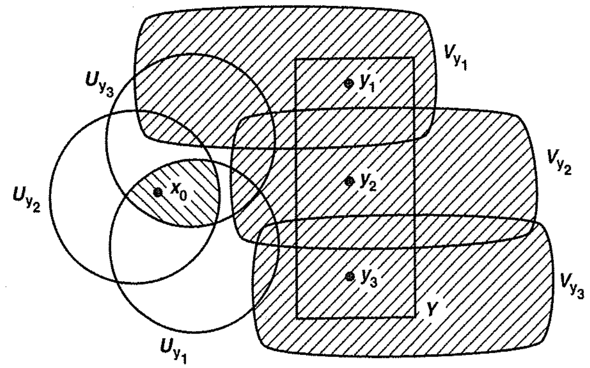

Figure 26.1

The statement we proved in the course of the preceding proof will be useful to us later, so we repeat it here for reference purposes:

Lemma 26.4. *If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.*

EXAMPLE 5. Once we prove that the interval $[a, b]$ in $\mathbb{R}$ is compact, it follows from Theorem 26.2 that any closed subspace of $[a, b]$ is compact. On the other hand, it follows from Theorem 26.3 that the intervals $(a, b]$ and (a, b) in $\mathbb{R}$ cannot be compact (which we knew already) because they are not closed in the Hausdorff space $\mathbb{R}$.

EXAMPLE 6. One needs the Hausdorff condition in the hypothesis of Theorem 26.3. Consider, for example, the finite complement topology on the real line. The only proper subsets of $\mathbb{R}$ that are closed in this topology are the finite sets. But *every* subset of $\mathbb{R}$ is compact in this topology, as you can check.

Theorem 26.5. *The image of a compact space under a continuous map is compact.*

Proof. Let $f : X \to Y$ be continuous; let X be compact. Let $\mathcal{A}$ be a covering of the set $f(X)$ by sets open in Y. The collection

$$\{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is a collection of sets covering X; these sets are open in X because f is continuous. Hence finitely many of them, say

$$f^{-1}(A_1), \dots, f^{-1}(A_n),$$

cover X. Then the sets $A_1, \dots, A_n$ cover $f(X)$. ∎

One important use of the preceding theorem is as a tool for verifying that a map is a homeomorphism:

Theorem 26.6. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. We shall prove that images of closed sets of X under f are closed in Y; this will prove continuity of the map f^{-1}. If A is closed in X, then A is compact, by Theorem 26.2. Therefore, by the theorem just proved, $f(A)$ is compact. Since Y is Hausdorff, $f(A)$ is closed in Y, by Theorem 26.3. ∎

Theorem 26.7. *The product of finitely many compact spaces is compact.*

Proof. We shall prove that the product of two compact spaces is compact; the theorem follows by induction for any finite product.

Step 1. Suppose that we are given spaces X and Y, with Y compact. Suppose that x_0 is a point of X, and N is an open set of $X \times Y$ containing the "slice" $x_0 \times Y$ of $X \times Y$. We prove the following:

There is a neighborhood W of x_0 in X such that N contains the entire set $W \times Y$.

The set $W \times Y$ is often called a ***tube*** about $x_0 \times Y$.

First let us cover $x_0 \times Y$ by basis elements $U \times V$ (for the topology of $X \times Y$) lying in N. The space $x_0 \times Y$ is compact, being homeomorphic to Y. Therefore, we can cover $x_0 \times Y$ by finitely many such basis elements

$$U_1 \times V_1, \dots, U_n \times V_n.$$

(We assume that each of the basis elements $U_i \times V_i$ actually intersects $x_0 \times Y$, since otherwise that basis element would be superfluous; we could discard it from the finite collection and still have a covering of $x_0 \times Y$.) Define

$$W = U_1 \cap \dots \cap U_n.$$

The set W is open, and it contains x_0 because each set $U_i \times V_i$ intersects $x_0 \times Y$.

We assert that the sets $U_i \times V_i$, which were chosen to cover the slice $x_0 \times Y$, actually cover the tube $W \times Y$. Let $x \times y$ be a point of $W \times Y$. Consider the point $x_0 \times y$ of the slice $x_0 \times Y$ having the same y-coordinate as this point. Now $x_0 \times y$ belongs to $U_i \times V_i$ for some i, so that $y \in V_i$. But $x \in U_j$ for *every* j (because $x \in W$). Therefore, we have $x \times y \in U_i \times V_i$, as desired.

Since all the sets $U_i \times V_i$ lie in N, and since they cover $W \times Y$, the tube $W \times Y$ lies in N also. See Figure 26.2.

Step 2. Now we prove the theorem. Let X and Y be compact spaces. Let $\mathcal{A}$ be an open covering of $X \times Y$. Given $x_0 \in X$, the slice $x_0 \times Y$ is compact and may therefore be covered by finitely many elements $A_1, \dots, A_m$ of $\mathcal{A}$. Their union $N = A_1 \cup \dots \cup A_m$ is an open set containing $x_0 \times Y$; by Step 1, the open set N contains

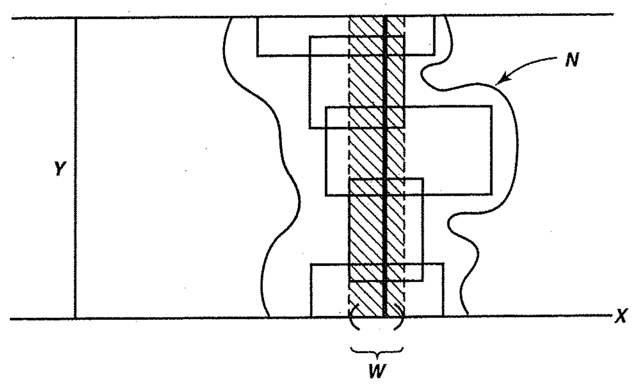

Figure 26.2

a tube $W \times Y$ about $x_0 \times Y$, where W is open in X. Then $W \times Y$ is covered by finitely many elements $A_1, \dots, A_m$ of $\mathcal{A}$.

Thus, for each x in X, we can choose a neighborhood W_x of x such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$. The collection of all the neighborhoods W_x is an open covering of X; therefore by compactness of X, there exists a finite subcollection

$$\{W_1, \dots, W_k\}$$

covering X. The union of the tubes

$$W_1 \times Y, \dots, W_k \times Y$$

is all of $X \times Y$; since each may be covered by finitely many elements of $\mathcal{A}$, so may $X \times Y$ be covered. ■

The statement proved in Step 1 of the preceding proof will be useful to us later, so we repeat it here as a lemma, for reference purposes:

Lemma 26.8 (The tube lemma). *Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.*

EXAMPLE 7. The tube lemma is certainly not true if Y is not compact. For example, let Y be the y-axis in $\mathbb{R}^2$, and let

$$N = \{x \times y; |x| < 1/(y^2 + 1)\}.$$

Then N is an open set containing the set $0 \times \mathbb{R}$, but it contains no tube about $0 \times \mathbb{R}$. It is illustrated in Figure 26.3.

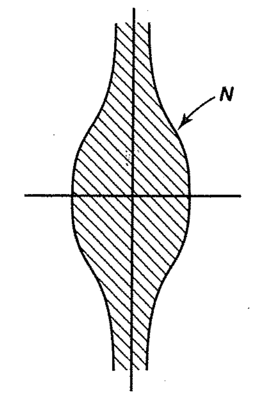

Figure 26.3

There is an obvious question to ask at this point. *Is the product of infinitely many compact spaces compact*? One would hope that the answer is "yes," and in fact it is. The result is important (and difficult) enough to be called by the name of the man who proved it; it is called the *Tychonoff theorem.*

In proving the fact that a cartesian product of connected spaces is connected, one proves it first for finite products and derives the general case from that. In proving that cartesian products of compact spaces are compact, however, there is no way to go directly from finite products to infinite ones. The infinite case demands a new approach, and the proof is a difficult one. Because of its difficulty, and also to avoid losing the main thread of our discussion in this chapter, we have decided to postpone it until later. However, you can study it now if you wish; the section in which it is proved (§37) can be studied immediately after this section without causing any disruption in continuity.

There is one final criterion for a space to be compact, a criterion that is formulated in terms of closed sets rather than open sets. It does not look very natural nor very useful at first glance, but it in fact proves to be useful on a number of occasions. First we make a definition.

Definition. A collection $\mathcal{C}$ of subsets of X is said to have the ***finite intersection property*** if for every finite subcollection

$$\{C_1, \dots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \dots \cap C_n$ is nonempty.

Theorem 26.9. *Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.*

Proof. Given a collection $\mathcal{A}$ of subsets of X, let

$$\mathcal{C} = \{X - A \mid A \in \mathcal{A}\}$$

be the collection of their complements. Then the following statements hold:

(1) $\mathcal{A}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.

(2) The collection $\mathcal{A}$ covers X if and only if the intersection $\bigcap_{C\in\mathcal{C}} C$ of all the elements of $\mathcal{C}$ is empty.

(3) The finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ covers X if and only if the intersection of the corresponding elements $C_i = X - A_i$ of $\mathcal{C}$ is empty.

The first statement is trivial, while the second and third follow from DeMorgan's law:

$$X - \left(\bigcup_{\alpha\in J} A_\alpha\right) = \bigcap_{\alpha\in J}(X - A_\alpha).$$

The proof of the theorem now proceeds in two easy steps: taking the *contrapositive* (of the theorem), and then the *complement* (of the sets)!

The statement that X is compact is equivalent to saying: "Given any collection $\mathcal{A}$ of open subsets of X, if $\mathcal{A}$ covers X, then some finite subcollection of $\mathcal{A}$ covers X." This statement is equivalent to its contrapositive, which is the following: "Given any collection $\mathcal{A}$ of open sets, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X." Letting $\mathcal{C}$ be, as earlier, the collection $\{X - A \mid A \in \mathcal{A}\}$ and applying (1)–(3), we see that this statement is in turn equivalent to the following: "Given any collection $\mathcal{C}$ of closed sets, if every finite intersection of elements of $\mathcal{C}$ is nonempty, then the intersection of all the elements of $\mathcal{C}$ is nonempty." This is just the condition of our theorem. ∎

A special case of this theorem occurs when we have a ***nested sequence*** $C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \ldots$ of closed sets in a compact space X. If each of the sets C_n is nonempty, then the collection $\mathcal{C} = \{C_n\}_{n\in\mathbb{Z}_+}$ automatically has the finite intersection property. Then the intersection

$$\bigcap_{n\in\mathbb{Z}_+} C_n$$

is nonempty.

We shall use the closed set criterion for compactness in the next section to prove the uncountability of the set of real numbers, in Chapter 5 when we prove the Tychonoff theorem, and again in Chapter 8 when we prove the Baire category theorem.

Exercises

1. (a) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X; suppose that $\mathcal{T}' \supset \mathcal{T}$. What does compactness of X under one of these topologies imply about compactness under the other?

(b) Show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T}$ and $\mathcal{T}'$ are equal or they are not comparable.

2. (a) Show that in the finite complement topology on $\mathbb{R}$, every subspace is compact.

(b) If $\mathbb{R}$ has the topology consisting of all sets A such that $\mathbb{R} - A$ is either countable or all of $\mathbb{R}$, is $[0, 1]$ a compact subspace?

3. Show that a finite union of compact subspaces of X is compact.

4. Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed bounded subspace is compact.

5. Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.

6. Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map (that is, f carries closed sets to closed sets).

7. Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

8. *Theorem. Let $f : X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the **graph** of f,*

$$G_f = \{x \times f(x) \mid x \in X\},$$

is closed in $X \times Y$. [*Hint:* If G_f is closed and V is a neighborhood of $f(x_0)$, then the intersection of G_f and $X \times (Y - V)$ is closed. Apply Exercise 7.]

9. Generalize the tube lemma as follows:

Theorem. Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that

$$A \times B \subset U \times V \subset N.$$

10. (a) Prove the following partial converse to the uniform limit theorem:

Theorem. Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions, with $f_n(x) \to f(x)$ for each $x \in X$. If f is continuous, and if the sequence f_n is monotone increasing, and if X is compact, then the convergence is uniform.

[We say that f_n is *monotone increasing* if $f_n(x) \le f_{n+1}(x)$ for all n and x.]

(b) Give examples to show that this theorem fails if you delete the requirement that X be compact, or if you delete the requirement that the sequence be monotone. [*Hint:* See the exercises of §21.]

11. *Theorem. Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then*

$$Y = \bigcap_{A\in\mathcal{A}} A$$

is connected. [*Hint:* If $C \cup D$ is a separation of Y, choose disjoint open sets U and V of X containing C and D, respectively, and show that

$$\bigcap_{A\in\mathcal{A}} (A - (U \cup V))$$

is not empty.]

12. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact, for each $y \in Y$. (Such a map is called a ***perfect map***.) Show that if Y is compact, then X is compact. [*Hint:* If U is an open set containing $p^{-1}(\{y\})$, there is a neighborhood W of y such that $p^{-1}(W)$ is contained in U.]

13. Let G be a topological group.

(a) Let A and B be subspaces of G. If A is closed and B is compact, show $A \cdot B$ is closed. [*Hint:* If c is not in $A \cdot B$, find a neighborhood W of c such that $W \cdot B^{-1}$ is disjoint from A.]

(b) Let H be a subgroup of G; let $p : G \to G/H$ be the quotient map. If H is compact, show that p is a closed map.

(c) Let H be a compact subgroup of G. Show that if G/H is compact, then G is compact.

§27 Compact Subspaces of the Real Line

The theorems of the preceding section enable us to construct new compact spaces from existing ones, but in order to get very far we have to find some compact spaces to start with. The natural place to begin is the real line; we shall prove that every closed interval in $\mathbb{R}$ is compact. Applications include the extreme value theorem and the uniform continuity theorem of calculus, suitably generalized. We also give a characterization of all compact subspaces of $\mathbb{R}^n$, and a proof of the uncountability of the set of real numbers.

It turns out that in order to prove every closed interval in $\mathbb{R}$ is compact, we need only *one* of the order properties of the real line—the least upper bound property. We shall prove the theorem using only this hypothesis; then it will apply not only to the real line, but to well-ordered sets and other ordered sets as well.

Theorem 27.1. *Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.*

Proof. *Step 1.* Given $a < b$, let $\mathcal{A}$ be a covering of $[a, b]$ by sets open in $[a, b]$ in the subspace topology (which is the same as the order topology). We wish to prove the existence of a finite subcollection of $\mathcal{A}$ covering $[a, b]$. First we prove the following: If x is a point of $[a, b]$ different from b, then there is a point $y > x$ of $[a, b]$ such that the interval $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.

If x has an immediate successor in X, let y be this immediate successor. Then $[x, y]$ consists of the two points x and y, so that it can be covered by at most two elements of $\mathcal{A}$. If x has no immediate successor in X, choose an element A of $\mathcal{A}$ containing x. Because $x \neq b$ and A is open, A contains an interval of the form $[x, c)$, for some c in $[a, b]$. Choose a point y in (x, c); then the interval $[x, y]$ is covered by the single element A of $\mathcal{A}$.

Step 2. Let C be the set of all points $y > a$ of $[a, b]$ such that the interval $[a, y]$ can be covered by finitely many elements of $\mathcal{A}$. Applying Step 1 to the case $x = a$, we see that there exists at least one such y, so C is not empty. Let c be the least upper bound of the set C; then $a < c \leq b$.

Step 3. We show that c belongs to C; that is, we show that the interval $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. Choose an element A of $\mathcal{A}$ containing c; since A is open, it contains an interval of the form $(d, c]$ for some d in $[a, b]$. If c is not in C, there must be a point z of C lying in the interval (d, c), because otherwise d would be a smaller upper bound on C than c. See Figure 27.1. Since z is in C, the interval $[a, z]$ can be covered by finitely many, say n, elements of $\mathcal{A}$. Now $[z, c]$ lies in the single element A of $\mathcal{A}$, hence $[a, c] = [a, z] \cup [z, c]$ can be covered by $n + 1$ elements of $\mathcal{A}$. Thus c is in C, contrary to assumption.

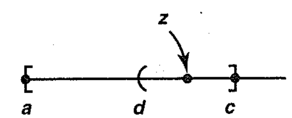

Figure 27.1

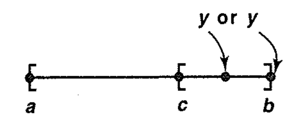

Figure 27.2

Step 4. Finally, we show that $c = b$, and our theorem is proved. Suppose that $c < b$. Applying Step 1 to the case $x = c$, we conclude that there exists a point $y > c$ of $[a, b]$ such that the interval $[c, y]$ can be covered by finitely many elements of $\mathcal{A}$. See Figure 27.2. We proved in Step 3 that c is in C, so $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. Therefore, the interval

$$[a, y] = [a, c] \cup [c, y]$$

can also be covered by finitely many elements of $\mathcal{A}$. This means that y is in C, contradicting the fact that c is an upper bound on C. ∎

Corollary 27.2. *Every closed interval in $\mathbb{R}$ is compact.*

Now we characterize the compact subspaces of $\mathbb{R}^n$:

Theorem 27.3. *A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d or the square metric ρ.*

Proof. It will suffice to consider only the metric ρ; the inequalities

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$$

imply that A is bounded under d if and only if it is bounded under ρ.

Suppose that A is compact. Then, by Theorem 26.3, it is closed. Consider the collection of open sets

$$\{B_\rho(\mathbf{0}, m) \mid m \in \mathbb{Z}_+\},$$

whose union is all of $\mathbb{R}^n$. Some finite subcollection covers A. It follows that $A \subset B_\rho(\mathbf{0}, M)$ for some M. Therefore, for any two points x and y of A, we have $\rho(x, y) \leq 2M$. Thus A is bounded under ρ.

Conversely, suppose that A is closed and bounded under ρ; suppose that $\rho(x, y) \leq N$ for every pair x, y of points of A. Choose a point x_0 of A, and let $\rho(x_0, \mathbf{0}) = b$. The triangle inequality implies that $\rho(x, \mathbf{0}) \leq N + b$ for every x in A. If $P = N + b$, then A is a subset of the cube $[-P, P]^n$, which is compact. Being closed, A is also compact. ∎

Students often remember this theorem as stating that the collection of compact sets in a *metric space* equals the collection of closed and bounded sets. This statement is clearly ridiculous as it stands, because the question as to which sets are bounded depends for its answer on the metric, whereas which sets are compact depends only on the topology of the space.

EXAMPLE 1. The unit sphere S^{n-1} and the closed unit ball B^n in $\mathbb{R}^n$ are compact because they are closed and bounded. The set

$$A = \{x \times (1/x) \mid 0 < x \leq 1\}$$

is closed in $\mathbb{R}^2$, but it is not compact because it is not bounded. The set

$$S = \{x \times (\sin(1/x)) \mid 0 < x \leq 1\}$$

is bounded in $\mathbb{R}^2$, but it is not compact because it is not closed.

Now we prove the extreme value theorem of calculus, in suitably generalized form.

Theorem 27.4 (Extreme value theorem). *Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

The extreme value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

Proof. Since f is continuous and X is compact, the set $A = f(X)$ is compact. We show that A has a largest element M and a smallest element m. Then since m and M belong to A, we must have $m = f(c)$ and $M = f(d)$ for some points c and d of X.

If A has no largest element, then the collection

$$\{(-\infty, a) \mid a \in A\}$$

forms an open covering of A. Since A is compact, some finite subcollection

$$\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$$

covers A. If a_i is the largest of the elements $a_1, \ldots a_n$, then a_i belongs to none of these sets, contrary to the fact that they cover A.

A similar argument shows that A has a smallest element. ∎

Now we prove the uniform continuity theorem of calculus. In the process, we are led to introduce a new notion that will prove to be surprisingly useful, that of a *Lebesgue number* for an open covering of a metric space. First, a preliminary notion:

Definition. Let (X, d) be a metric space; let A be a nonempty subset of X. For each $x \in X$, we define the ***distance from x to A*** by the equation

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

It is easy to show that for fixed A, the function $d(x, A)$ is a continuous function of x: Given $x, y \in X$, one has the inequalities

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a),$$

for each $a \in A$. It follows that

$$d(x, A) - d(x, y) \leq \inf d(y, a) = d(y, A),$$

so that

$$d(x, A) - d(y, A) \leq d(x, y).$$

The same inequality holds with x and y interchanged; continuity of the function $d(x, A)$ follows.

Now we introduce the notion of Lebesgue number. Recall that the *diameter* of a bounded subset A of a metric space (X, d) is the number

$$\sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}.$$

Lemma 27.5 (The Lebesgue number lemma). *Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it.*

The number δ is called a ***Lebesgue number*** for the covering $\mathcal{A}$.

Proof. Let $\mathcal{A}$ be an open covering of X. If X itself is an element of $\mathcal{A}$, then any positive number is a Lebesgue number for $\mathcal{A}$. So assume X is not an element of $\mathcal{A}$.

Choose a finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ that covers X. For each i, set $C_i = X - A_i$, and define $f : X \to \mathbb{R}$ by letting $f(x)$ be the average of the numbers $d(x, C_i)$. That is,

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, C_i).$$

We show that $f(x) > 0$ for all x. Given $x \in X$, choose i so that $x \in A_i$. Then choose ϵ so the ϵ-neighborhood of x lies in A_i. Then $d(x, C_i) \geq \epsilon$, so that $f(x) \geq \epsilon/n$.

Since f is continuous, it has a minimum value δ; we show that δ is our required Lebesgue number. Let B be a subset of X of diameter less than δ. Choose a point x_0 of B; then B lies in the δ-neighborhood of x_0. Now

$$\delta \leq f(x_0) \leq d(x_0, C_m),$$

where $d(x_0, C_m)$ is the largest of the numbers $d(x_0, C_i)$. Then the δ-neighborhood of x_0 is contained in the element $A_m = X - C_m$ of the covering $\mathcal{A}$. ■

Definition. A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be ***uniformly continuous*** if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

$$d_X(x_0, x_1) < \delta \Longrightarrow d_Y(f(x_0), f(x_1)) < \epsilon.$$

Theorem 27.6 (Uniform continuity theorem). *Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.*

Proof. Given $\epsilon > 0$, take the open covering of Y by balls $B(y, \epsilon/2)$ of radius $\epsilon/2$. Let $\mathcal{A}$ be the open covering of X by the inverse images of these balls under f. Choose δ to be a Lebesgue number for the covering $\mathcal{A}$. Then if x_1 and x_2 are two points of X such that $d_X(x_1, x_2) < \delta$, the two-point set $\{x_1, x_2\}$ has diameter less than δ, so that its image $\{f(x_1), f(x_2)\}$ lies in some ball $B(y, \epsilon/2)$. Then $d_Y(f(x_1), f(x_2)) < \epsilon$, as desired. ■

Finally, we prove that the real numbers are uncountable. The interesting thing about this proof is that it involves no algebra at all—no decimal or binary expansions of real numbers or the like—just the order properties of $\mathbb{R}$.

Definition. If X is a space, a point x of X is said to be an ***isolated point*** of X if the one-point set $\{x\}$ is open in X.

Theorem 27.7. *Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. Step 1. We show first that given any nonempty open set U of X and any point x of X, there exists a nonempty open set V contained in U such that $x \notin \bar{V}$.

Choose a point y of U different from x; this is possible if x is in U because x is not an isolated point of X and it is possible if x is not in U simply because U is nonempty. Now choose disjoint open sets W_1 and W_2 about x and y, respectively. Then the set $V = W_2 \cap U$ is the desired open set; it is contained in U, it is nonempty because it contains y, and its closure does not contain x. See Figure 27.3.

Step 2. We show that given $f : \mathbb{Z}_+ \to X$, the function f is not surjective. It follows that X is uncountable.

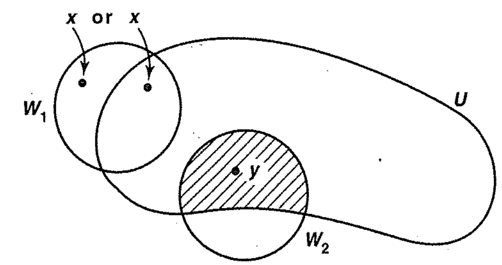

Figure 27.3

Let $x_n = f(n)$. Apply Step 1 to the nonempty open set $U = X$ to choose a nonempty open set $V_1 \subset X$ such that $\bar{V}_1$ does not contain x_1. In general, given V_{n-1} open and nonempty, choose V_n to be a nonempty open set such that $V_n \subset V_{n-1}$ and $\bar{V}_n$ does not contain x_n. Consider the nested sequence

$$\bar{V}_1 \supset \bar{V}_2 \supset \cdots$$

of nonempty closed sets of X. Because X is compact, there is a point $x \in \bigcap \bar{V}_n$, by Theorem 26.9. Now x cannot equal x_n for any n, since x belongs to $\bar{V}_n$ and x_n does not. ■

Corollary 27.8. *Every closed interval in $\mathbb{R}$ is uncountable.*

Exercises

1. Prove that if X is an ordered set in which every closed interval is compact, then X has the least upper bound property.
2. Let X be a metric space with metric d; let $A \subset X$ be nonempty.
   (a) Show that $d(x, A) = 0$ if and only if $x \in \bar{A}$.
   (b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.
   (c) Define the *ϵ-neighborhood* of A in X to be the set
   $$U(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}.$$
   Show that $U(A, \epsilon)$ equals the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.
   (d) Assume that A is compact; let U be an open set containing A. Show that some ϵ-neighborhood of A is contained in U.
   (e) Show the result in (d) need not hold if A is closed but not compact.
3. Recall that $\mathbb{R}_K$ denotes $\mathbb{R}$ in the K-topology.
   (a) Show that $[0, 1]$ is not compact as a subspace of $\mathbb{R}_K$.

(b) Show that $\mathbb{R}_K$ is connected. [*Hint:* $(-\infty, 0)$ and $(0, \infty)$ inherit their usual topologies as subspaces of $\mathbb{R}_K$.]

(c) Show that $\mathbb{R}_K$ is not path connected.

4. Show that a connected metric space having more than one point is uncountable.

5. Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\bigcup A_n$ has empty interior in X. [*Hint:* Imitate the proof of Theorem 27.7.]

This is a special case of the *Baire category theorem*, which we shall study in Chapter 8.

6. Let A_0 be the closed interval $[0, 1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its "middle third" $(\frac{1}{3}, \frac{2}{3})$. Let A_2 be the set obtained from A_1 by deleting its "middle thirds" $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, define A_n by the equation

$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

The intersection

$$C = \bigcap_{n \in \mathbb{Z}_+} A_n$$

is called the ***Cantor set***; it is a subspace of $[0, 1]$.

(a) Show that C is totally disconnected.

(b) Show that C is compact.

(c) Show that each set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$; and show that the end points of these intervals lie in C.

(d) Show that C has no isolated points.

(e) Conclude that C is uncountable.

§28 Limit Point Compactness

As indicated when we first mentioned compact sets, there are other formulations of the notion of compactness that are frequently useful. In this section we introduce one of them. Weaker in general than compactness, it coincides with compactness for metrizable spaces.

Definition. A space X is said to be ***limit point compact*** if every infinite subset of X has a limit point.

In some ways this property is more natural and intuitive than that of compactness. In the early days of topology, it was given the name "compactness," while the open covering formulation was called "bicompactness." Later, the word "compact" was shifted to apply to the open covering definition, leaving this one to search for a new name. It still has not found a name on which everyone agrees. On historical grounds, some call it "Fréchet compactness"; others call it the "Bolzano-Weierstrass property." We have invented the term "limit point compactness." It seems as good a term as any; at least it describes what the property is about.

Theorem 28.1. *Compactness implies limit point compactness, but not conversely.*

Proof. Let X be a compact space. Given a subset A of X, we wish to prove that if A is infinite, then A has a limit point. We prove the contrapositive—if A has no limit point, then A must be finite.

So suppose A has no limit point. Then A contains all its limit points, so that A is closed. Furthermore, for each $a \in A$ we can choose a neighborhood U_a of a such that U_a intersects A in the point a alone. The space X is covered by the open set $X - A$ and the open sets U_a; being compact, it can be covered by finitely many of these sets. Since $X - A$ does not intersect A, and each set U_a contains only one point of A, the set A must be finite. ∎

EXAMPLE 1. Let Y consist of two points; give Y the topology consisting of Y and the empty set. Then the space $X = \mathbb{Z}_+ \times Y$ is limit point compact, for *every* nonempty subset of X has a limit point. It is not compact, for the covering of X by the open sets $U_n = \{n\} \times Y$ has no finite subcollection covering X.

EXAMPLE 2. Here is a less trivial example. Consider the minimal uncountable well-ordered set S_Ω, in the order topology. The space S_Ω is not compact, since it has no largest element. However, it is limit point compact: Let A be an infinite subset of S_Ω. Choose a subset B of A that is countably infinite. Being countable, the set B has an upper bound b in S_Ω; then B is a subset of the interval $[a_0, b]$ of S_Ω, where a_0 is the smallest element of S_Ω. Since S_Ω has the least upper bound property, the interval $[a_0, b]$ is compact. By the preceding theorem, B has a limit point x in $[a_0, b]$. The point x is also a limit point of A. Thus S_Ω is limit point compact.

We now show these two versions of compactness coincide for metrizable spaces; for this purpose, we introduce yet another version of compactness called *sequential compactness*. This result will be used in Chapter 7.

Definition. Let X be a topological space. If (x_n) is a sequence of points of X, and if

$$n_1 < n_2 < \cdots < n_i < \cdots$$

is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a ***subsequence*** of the sequence (x_n). The space X is said to be ***sequentially compact*** if every sequence of points of X has a convergent subsequence.

***Theorem 28.2.** *Let X be a metrizable space. Then the following are equivalent:*

(1) *X is compact.*

(2) *X is limit point compact.*

(3) *X is sequentially compact.*

Proof. We have already proved that (1) $\Rightarrow$ (2). To show that (2) $\Rightarrow$ (3), assume that X is limit point compact. Given a sequence (x_n) of points of X, consider the set $A = \{x_n \mid n \in \mathbb{Z}_+\}$. If the set A is finite, then there is a point x such that $x = x_n$ for infinitely many values of n. In this case, the sequence (x_n) has a subsequence that is constant, and therefore converges trivially. On the other hand, if A is infinite, then A has a limit point x. We define a subsequence of (x_n) converging to x as follows: First choose n_1 so that

$$x_{n_1} \in B(x, 1).$$

Then suppose that the positive integer n_{i-1} is given. Because the ball $B(x, 1/i)$ intersects A in infinitely many points, we can choose an index $n_i > n_{i-1}$ such that

$$x_{n_i} \in B(x, 1/i).$$

Then the subsequence $x_{n_1}, x_{n_2}, \ldots$ converges to x.

Finally, we show that (3) $\Rightarrow$ (1). This is the hardest part of the proof.

First, we show that if X is sequentially compact, then the Lebesgue number lemma holds for X. (This would follow from compactness, but compactness is what we are trying to prove!) Let $\mathcal{A}$ be an open covering of X. We assume that there is no $\delta > 0$ such that each set of diameter less than δ has an element of $\mathcal{A}$ containing it, and derive a contradiction.

Our assumption implies in particular that for each positive integer n, there exists a set of diameter less than $1/n$ that is not contained in any element of $\mathcal{A}$; let C_n be such a set. Choose a point $x_n \in C_n$, for each n. By hypothesis, some subsequence (x_{n_i}) of the sequence (x_n) converges, say to the point a. Now a belongs to some element A of the collection $\mathcal{A}$; because A is open, we may choose an $\epsilon > 0$ such that $B(a, \epsilon) \subset A$. If i is large enough that $1/n_i < \epsilon/2$, then the set C_{n_i} lies in the $\epsilon/2$-neighborhood of x_{n_i}; if i is also chosen large enough that $d(x_{n_i}, a) < \epsilon/2$, then C_{n_i} lies in the ϵ-neighborhood of a. But this means that $C_{n_i} \subset A$, contrary to hypothesis.

Second, we show that if X is sequentially compact, then given $\epsilon > 0$, there exists a finite covering of X by open ϵ-balls. Once again, we proceed by contradiction. Assume that there exists an $\epsilon > 0$ such that X cannot be covered by finitely many ϵ-balls. Construct a sequence of points x_n of X as follows: First, choose x_1 to be any point of X. Noting that the ball $B(x_1, \epsilon)$ is not all of X (otherwise X could be covered by a single ϵ-ball), choose x_2 to be a point of X not in $B(x_1, \epsilon)$. In general, given $x_1, \ldots, x_n$, choose x_{n+1} to be a point not in the union

$$B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon),$$

using the fact that these balls do not cover X. Note that by construction $d(x_{n+1}, x_i) \geq \epsilon$ for $i = 1, \ldots, n$. Therefore, the sequence (x_n) can have no convergent subsequence; in fact, any ball of radius $\epsilon/2$ can contain x_n for at most *one* value of n.

Finally, we show that if X is sequentially compact, then X is compact. Let $\mathcal{A}$ be an open covering of X. Because X is sequentially compact, the open covering $\mathcal{A}$ has a Lebesgue number δ. Let $\epsilon = \delta/3$; use sequential compactness of X to find a finite covering of X by open ϵ-balls. Each of these balls has diameter at most $2\delta/3$, so it lies in an element of $\mathcal{A}$. Choosing one such element of $\mathcal{A}$ for each of these ϵ-balls, we obtain a finite subcollection of $\mathcal{A}$ that covers X. ∎

EXAMPLE 3. Recall that $\bar{S}_\Omega$ denotes the minimal uncountable well-ordered set S_Ω with the point Ω adjoined. (In the order topology, Ω is a limit point of S_Ω, which is why we introduced the notation $\bar{S}_\Omega$ for $S_\Omega \cup \{\Omega\}$, back in §10.) It is easy to see that the space $\bar{S}_\Omega$ is not metrizable, for it does not satisfy the sequence lemma: The point Ω is a limit point of S_Ω; but it is not the limit of a sequence of points of S_Ω, for any sequence of points of S_Ω has an upper bound in S_Ω. The space S_Ω, on the other hand, does satisfy the sequence lemma, as you can readily check. Nevertheless, S_Ω is not metrizable, for it is limit point compact but not compact.

Exercises

1. Give $[0, 1]^\omega$ the uniform topology. Find an infinite subset of this space that has no limit point.

2. Show that $[0, 1]$ is not limit point compact as a subspace of $\mathbb{R}_\ell$.

3. Let X be limit point compact.
  (a) If $f : X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact?
  (b) If A is a closed subset of X, does it follow that A is limit point compact?
  (c) If X is a subspace of the Hausdorff space Z, does it follow that X is closed in Z?

We comment that it is not in general true that the product of two limit point compact spaces is limit point compact, even if the Hausdorff condition is assumed. But the examples are fairly sophisticated. See [S-S], Example 112.

4. A space X is said to be ***countably compact*** if every countable open covering of X contains a finite subcollection that covers X. Show that for a T_1 space X, countable compactness is equivalent to limit point compactness. [*Hint:* If no finite subcollection of U_n covers X, choose $x_n \notin U_1 \cup \cdots \cup U_n$, for each n.]

5. Show that X is countably compact if and only if every nested sequence $C_1 \supset C_2 \supset \cdots$ of closed nonempty sets of X has a nonempty intersection.

6. Let (X, d) be a metric space. If $f : X \to X$ satisfies the condition

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$, then f is called an ***isometry*** of X. Show that if f is an isometry and X is compact, then f is bijective and hence a homeomorphism. [*Hint:* If $a \notin f(X)$, choose ϵ so that the ϵ-neighborhood of a is disjoint from $f(X)$. Set $x_1 = a$, and $x_{n+1} = f(x_n)$ in general. Show that $d(x_n, x_m) \geq \epsilon$ for $n \neq m$.]

7. Let (X, d) be a metric space. If f satisfies the condition

$$d(f(x), f(y)) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$, then f is called a ***shrinking map***. If there is a number $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$, then f is called a ***contraction***. A ***fixed point*** of f is a point x such that $f(x) = x$.

(a) If f is a contraction and X is compact, show f has a unique fixed point. [*Hint:* Define $f^1 = f$ and $f^{n+1} = f \circ f^n$. Consider the intersection A of the sets $A_n = f^n(X)$.]

(b) Show more generally that if f is a shrinking map and X is compact, then f has a unique fixed point. [*Hint:* Let A be as before. Given $x \in A$, choose x_n so that $x = f^{n+1}(x_n)$. If a is the limit of some subsequence of the sequence $y_n = f^n(x_n)$, show that $a \in A$ and $f(a) = x$. Conclude that $A = f(A)$, so that diam $A = 0$.]

(c) Let $X = [0, 1]$. Show that $f(x) = x - x^2/2$ maps X into X and is a shrinking map that is not a contraction. [*Hint:* Use the mean-value theorem of calculus.]

(d) The result in (a) holds if X is a complete metric space, such as $\mathbb{R}$; see the exercises of §43. The result in (b) does not: Show that the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = [x + (x^2 + 1)^{1/2}]/2$ is a shrinking map that is not a contraction and has no fixed point.

§29 Local Compactness

In this section we study the notion of local compactness, and we prove the basic theorem that any locally compact Hausdorff space can be imbedded in a certain compact Hausdorff space that is called its *one-point compactification*.

Definition. A space X is said to be ***locally compact at x*** if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said simply to be ***locally compact***.

Note that a compact space is automatically locally compact.

EXAMPLE 1. The real line $\mathbb{R}$ is locally compact. The point x lies in some interval (a, b), which in turn is contained in the compact subspace $[a, b]$. The subspace $\mathbb{Q}$ of rational numbers is not locally compact, as you can check.

EXAMPLE 2. The space $\mathbb{R}^n$ is locally compact; the point x lies in some basis element $(a_1, b_1) \times \cdots \times (a_n, b_n)$, which in turn lies in the compact subspace $[a_1, b_1] \times \cdots \times [a_n, b_n]$. The space $\mathbb{R}^\omega$ is not locally compact; *none* of its basis elements are contained in compact subspaces. For if

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \cdots \times \mathbb{R} \times \cdots$$

were contained in a compact subspace, then its closure

$$\bar{B} = [a_1, b_1] \times \cdots \times [a_n, b_n] \times \mathbb{R} \times \cdots$$

would be compact, which it is not.

EXAMPLE 3. Every simply ordered set X having the least upper bound property is locally compact: Given a basis element for X, it is contained in a closed interval in X, which is compact.

Two of the most well-behaved classes of spaces to deal with in mathematics are the metrizable spaces and the compact Hausdorff spaces. Such spaces have many useful properties, which one can use in proving theorems and making constructions and the like. If a given space is not of one of these types, the next best thing one can hope for is that it is a subspace of one of these spaces. Of course, a subspace of a metrizable space is itself metrizable, so one does not get any new spaces in this way. But a subspace of a compact Hausdorff space need not be compact. Thus arises the question: Under what conditions is a space homeomorphic with a subspace of a compact Hausdorff space? We give one answer here. We shall return to this question in Chapter 5 when we study compactifications in general.

Theorem 29.1. *Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions:*

(1) *X is a subspace of Y.*

(2) *The set $Y - X$ consists of a single point.*

(3) *Y is a compact Hausdorff space.*

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Proof. *Step 1.* We first verify uniqueness. Let Y and Y' be two spaces satisfying these conditions. Define $h : Y \to Y'$ by letting h map the single point p of $Y - X$ to the point q of $Y' - X$, and letting h equal the identity on X. We show that if U is open in Y, then $h(U)$ is open in Y'. Symmetry then implies that h is a homeomorphism.

First, consider the case where U does not contain p. Then $h(U) = U$. Since U is open in Y and is contained in X, it is open in X. Because X is open in Y', the set U is also open in Y', as desired.

Second, suppose that U contains p. Since $C = Y - U$ is closed in Y, it is compact as a subspace of Y. Because C is contained in X, it is a compact subspace of X. Then because X is a subspace of Y', the space C is also a compact subspace of Y'. Because Y' is Hausdorff, C is closed in Y', so that $h(U) = Y' - C$ is open in Y', as desired.

Step 2. Now we suppose X is locally compact Hausdorff and construct the space Y. Step 1 gives us an idea how to proceed. Let us take some object that is not a point of X, denote it by the symbol ∞ for convenience, and adjoin it to X, forming the set $Y = X \cup \{\infty\}$. Topologize Y by defining the collection of open sets of Y to consist of (1) all sets U that are open in X, and (2) all sets of the form $Y - C$, where C is a compact subspace of X.

We need to check that this collection is, in fact, a topology on Y. The empty set is a set of type (1), and the space Y is a set of type (2). Checking that the intersection of two open sets is open involves three cases:

$$\begin{aligned} U_1 \cap U_2 \quad & \text{is of type (1).} \\ (Y - C_1) \cap (Y - C_2) = Y - (C_1 \cup C_2) \quad & \text{is of type (2).} \\ U_1 \cap (Y - C_1) = U_1 \cap (X - C_1) \quad & \text{is of type (1),} \end{aligned}$$

because C_1 is closed in X. Similarly, one checks that the union of any collection of open sets is open:

$$\begin{aligned} \bigcup U_\alpha = U \quad & \text{is of type (1).} \\ \bigcup (Y - C_\beta) = Y - (\bigcap C_\beta) = Y - C \quad & \text{is of type (2).} \\ (\bigcup U_\alpha) \cup (\bigcup (Y - C_\beta)) = U \cup (Y - C) = Y - (C - U), & \end{aligned}$$

which is of type (2) because $C - U$ is a closed subspace of C and therefore compact.

Now we show that X is a subspace of Y. Given any open set of Y, we show its intersection with X is open in X. If U is of type (1), then $U \cap X = U$; if $Y - C$ is of type (2), then $(Y - C) \cap X = X - C$; both of these sets are open in X. Conversely, any set open in X is a set of type (1) and therefore open in Y by definition.

To show that Y is compact, let $\mathcal{A}$ be an open covering of Y. The collection $\mathcal{A}$ must contain an open set of type (2), say $Y - C$, since none of the open sets of type (1) contain the point ∞. Take all the members of $\mathcal{A}$ different from $Y - C$ and intersect them with X; they form a collection of open sets of X covering C. Because C is compact, finitely many of them cover C; the corresponding finite collection of elements of $\mathcal{A}$ will, along with the element $Y - C$, cover all of Y.

To show that Y is Hausdorff, let x and y be two points of Y. If both of them lie in X, there are disjoint sets U and V open in X containing them, respectively. On the other hand, if $x \in X$ and $y = \infty$, we can choose a compact set C in X containing a neighborhood U of x. Then U and $Y - C$ are disjoint neighborhoods of x and ∞, respectively, in Y.

Step 3. Finally, we prove the converse. Suppose a space Y satisfying conditions (1)–(3) exists. Then X is Hausdorff because it is a subspace of the Hausdorff space Y. Given $x \in X$, we show X is locally compact at x. Choose disjoint open sets U and V of Y containing x and the single point of $Y - X$, respectively. Then the set $C = Y - V$ is closed in Y, so it is a compact subspace of Y. Since C lies in X, it is also compact as a subspace of X; it contains the neighborhood U of x. ■

If X itself should happen to be compact, then the space Y of the preceding theorem is not very interesting, for it is obtained from X by adjoining a single isolated point. However, if X is not compact, then the point of $Y - X$ is a limit point of X, so that $\bar{X} = Y$.

Definition. If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be a ***compactification*** of X. If $Y - X$ equals a single point, then Y is called the ***one-point compactification*** of X.

We have shown that X has a one-point compactification Y if and only if X is a locally compact Hausdorff space that is not itself compact. We speak of Y as "the" one-point compactification because Y is uniquely determined up to a homeomorphism.

EXAMPLE 4. The one-point compactification of the real line $\mathbb{R}$ is homeomorphic with the circle, as you may readily check. Similarly, the one-point compactification of $\mathbb{R}^2$ is homeomorphic to the sphere S^2. If $\mathbb{R}^2$ is looked at as the space $\mathbb{C}$ of complex numbers, then $\mathbb{C} \cup \{\infty\}$ is called the *Riemann sphere*, or the *extended complex plane*.

In some ways our definition of local compactness is not very satisfying. Usually one says that a space X satisfies a given property "locally" if every $x \in X$ has "arbitrarily small" neighborhoods having the given property. Our definition of local compactness has nothing to do with "arbitrarily small" neighborhoods, so there is some question whether we should call it local compactness at all.

Here is another formulation of local compactness, one more truly "local" in nature; it is equivalent to our definition when X is Hausdorff.

Theorem 29.2. *Let X be a Hausdorff space. Then X is locally compact if and only if given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.*

Proof. Clearly this new formulation implies local compactness; the set $C = \bar{V}$ is the desired compact set containing a neighborhood of x. To prove the converse, suppose X is locally compact; let x be a point of X and let U be a neighborhood of x. Take the one-point compactification Y of X, and let C be the set $Y - U$. Then C is closed in Y, so that C is a compact subspace of Y. Apply Lemma 26.4 to choose disjoint open sets V and W containing x and C, respectively. Then the closure $\bar{V}$ of V in Y is compact; furthermore, $\bar{V}$ is disjoint from C, so that $\bar{V} \subset U$, as desired. ■

Corollary 29.3. *Let X be locally compact Hausdorff; let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.*

Proof. Suppose that A is closed in X. Given $x \in A$, let C be a compact subspace of X containing the neighborhood U of x in X. Then $C \cap A$ is closed in C and thus compact, and it contains the neighborhood $U \cap A$ of x in A. (We have not used the Hausdorff condition here.)

Suppose now that A is open in X. Given $x \in A$, we apply the preceding theorem to choose a neighborhood V of x in X such that $\bar{V}$ is compact and $\bar{V} \subset A$. Then $C = \bar{V}$ is a compact subspace of A containing the neighborhood V of x in A. ■

Corollary 29.4. *A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.*

Proof. This follows from Theorem 29.1 and Corollary 29.3. ■

Exercises

1. Show that the rationals $\mathbb{Q}$ are not locally compact.
2. Let $\{X_\alpha\}$ be an indexed family of nonempty spaces.
   (a) Show that if $\prod X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α.
   (b) Prove the converse, assuming the Tychonoff theorem.
3. Let X be a locally compact space. If $f : X \to Y$ is continuous, does it follow that $f(X)$ is locally compact? What if f is both continuous and open? Justify your answer.
4. Show that $[0, 1]^\omega$ is not locally compact in the uniform topology.
5. If $f : X_1 \to X_2$ is a homeomorphism of locally compact Hausdorff spaces, show f extends to a homeomorphism of their one-point compactifications.
6. Show that the one-point compactification of $\mathbb{R}$ is homeomorphic with the circle S^1.
7. Show that the one-point compactification of S_Ω is homeomorphic with $\bar{S}_\Omega$.
8. Show that the one-point compactification of $\mathbb{Z}_+$ is homeomorphic with the subspace $\{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$.
9. Show that if G is a locally compact topological group and H is a subgroup, then G/H is locally compact.
10. Show that if X is a Hausdorff space that is locally compact at the point x, then for each neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

*11. Prove the following:
   (a) *Lemma. If $p : X \to Y$ is a quotient map and if Z is a locally compact Hausdorff space, then the map*
   $$\pi = p \times i_Z : X \times Z \longrightarrow Y \times Z$$
   is a quotient map.
   [*Hint:* If $\pi^{-1}(A)$ is open and contains $x \times y$, choose open sets U_1 and V with $\bar{V}$ compact, such that $x \times y \in U_1 \times V$ and $U_1 \times \bar{V} \subset \pi^{-1}(A)$. Given $U_i \times \bar{V} \subset \pi^{-1}(A)$, use the tube lemma to choose an open set U_{i+1} containing $p^{-1}(p(U_i))$ such that $U_{i+1} \times \bar{V} \subset \pi^{-1}(A)$. Let $U = \bigcup U_i$; show that $U \times V$ is a saturated neighborhood of $x \times y$ that is contained in $\pi^{-1}(A)$.]
   An entirely different proof of this result will be outlined in the exercises of §46.
   (b) *Theorem. Let $p : A \to B$ and $q : C \to D$ be quotient maps. If B and C are locally compact Hausdorff spaces, then $p \times q : A \times C \to B \times D$ is a quotient map.*

*Supplementary Exercises: Nets

We have already seen that sequences are "adequate" to detect limit points, continuous functions, and compact sets in metrizable spaces. There is a generalization of the notion of sequence, called a *net*, that will do the same thing for an arbitrary topological space. We give the relevant definitions here, and leave the proofs as exercises. Recall that a relation $\preceq$ on a set A is called a *partial order* relation if the following conditions hold:

(1) $\alpha \preceq \alpha$ for all α.
(2) If $\alpha \preceq \beta$ and $\beta \preceq \alpha$, then $\alpha = \beta$.
(3) If $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$.

Now we make the following definition:

A ***directed set*** J is a set with a partial order $\preceq$ such that for each pair α, β of elements of J, there exists an element γ of J having the property that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

1. Show that the following are directed sets:
   (a) Any simply ordered set, under the relation $\leq$.
   (b) The collection of all subsets of a set S, partially ordered by inclusion (that is, $A \preceq B$ if $A \subset B$).
   (c) A collection $\mathcal{A}$ of subsets of S that is closed under finite intersections, partially ordered by reverse inclusion (that is $A \preceq B$ if $A \supset B$).
   (d) The collection of all closed subsets of a space X, partially ordered by inclusion.
2. A subset K of J is said to be ***cofinal*** in J if for each $\alpha \in J$, there exists $\beta \in K$ such that $\alpha \preceq \beta$. Show that if J is a directed set and K is cofinal in J, then K is a directed set.
3. Let X be a topological space. A ***net*** in X is a function f from a directed set J into X. If $\alpha \in J$, we usually denote $f(\alpha)$ by x_α. We denote the net f itself by the symbol $(x_\alpha)_{\alpha \in J}$, or merely by (x_α) if the index set is understood.
   The net (x_α) is said to ***converge*** to the point x of X (written $x_\alpha \to x$) if for each neighborhood U of x, there exists $\alpha \in J$ such that
   $$\alpha \preceq \beta \Longrightarrow x_\beta \in U.$$
   Show that these definitions reduce to familiar ones when $J = \mathbb{Z}_+$.
4. Suppose that
   $$(x_\alpha)_{\alpha \in J} \longrightarrow x \text{ in } X \quad \text{and} \quad (y_\alpha)_{\alpha \in J} \longrightarrow y \text{ in } Y.$$
   Show that $(x_\alpha \times y_\alpha) \longrightarrow x \times y$ in $X \times Y$.
5. Show that if X is Hausdorff, a net in X converges to at most one point.
6. *Theorem. Let $A \in X$. Then $x \in \bar{A}$ if and only if there is a net of points of A converging to x.*
   [*Hint:* To prove the implication $\Rightarrow$, take as index set the collection of all neighborhoods of x, partially ordered by reverse inclusion.]