Solution Key - Homework 1 - Topology

Pb. 1. One of the many possible examples:

Let $\mathcal{T}_1 = \{\emptyset, \{a,b\}, \{a,b,c,d\}\}$

$\mathcal{T}_2 = \{\emptyset, \{a,c\}, \{a,b,c,d\}\}$

Both $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on $X = \{a,b,c,d\}$.
(They contain $\emptyset$ and X and are closed under unions and intersections)

But $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{a,b\}, \{a,c\}, X\}$ is <u>not</u> a topology

since $\{a,b\} \cap \{a,c\} = \{a\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$
(also $\{a,b\} \cup \{a,c\} = \{a,b,c\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$)

q.e.d.

Further comment / problem

Given a set X and two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X, there is always a "smallest" topology $\mathcal{T}$ on X that contains $\mathcal{T}_1 \cup \mathcal{T}_2$. How do you define $\mathcal{T}$?

Pb. 2: We need to show that $\mathcal{B}$ satisfies the two conditions of Theorem 1.8.

$\mathbb{R} \subseteq \bigcup_{B \in \mathcal{B}} B$ since $\forall x \in \mathbb{R}$, $x \in (x-1, x+1)$ and $(x-1, x+1) \in \mathcal{B}$

As $\forall B \in \mathcal{B}$, $B \subseteq \mathbb{R}$, $\bigcup_{B \in \mathcal{B}} B \subseteq \mathbb{R}$

we have the 1^{st} condition $\bigcup_{B \in \mathcal{B}} B = \mathbb{R}$.

The second condition is automatically satisfied if we have that $\forall B_1, B_2 \in \mathcal{B} \Rightarrow B_1 \cap B_2 \in \mathcal{B}$.

But this is true in our case:

- if $B_1 = (a_1, b_1)$, $B_2 = (a_2, b_2)$ are open intervals then
$$B_1 \cap B_2 = \begin{cases} (\max(a_1, a_2), \min(b_1, b_2)) & \text{if } \max(a_1, a_2) < \min(b_1, b_2) \\ \emptyset & \text{otherwise} \end{cases}$$
so $B_1 \cap B_2$ is an open interval or the empty set

- if one (or both) of B_1 and B_2 are of the form $B_1 = I_1 \setminus A$, $B_2 = I_2 \setminus A$ with I_1, I_2 open intervals then $B_1 \cap B_2 = (I_1 \cap I_2) \setminus A$, so still an element of $\mathcal{B}$.

Thus $\mathcal{B}$ generates a topology on $\mathbb{R}$, let's call it $\mathcal{T}'$.

Pb. 2 - continuation

If $\mathcal{T}$ is the usual (Euclidean) topology on $\mathbb{R}$, we claim that $\mathcal{T}'$ is <u>strictly finer</u> than $\mathcal{T}$.

$\mathcal{T} \subseteq \mathcal{T}'$ because open intervals are a basis for the usual topology and open intervals are a subcollection of the basis $\mathcal{B}$ of new topology.

To see that $\mathcal{T} \neq \mathcal{T}'$ consider the set

$(-1,1) \setminus A$. Obviously $(-1,1) \setminus A \in \mathcal{T}'$,

but $(-1,1) \setminus A \notin \mathcal{T}$. $(*)$

pf of $(*)$ If $(-1,1) \setminus A \in \mathcal{T}$, as $0 \in (-1,1) \setminus A$ it follows that $\exists\, \varepsilon > 0$ s.t. $0 \in (-\varepsilon, \varepsilon) \subseteq (-1,1) \setminus A$

But given $\varepsilon > 0$, $\exists\, n \in \mathbb{N}^+$ s.t. $\frac{1}{n} < \varepsilon$

as, by definition, $\frac{1}{n} \in A$. $\lrcorner$

Pb. 3 : Instead of a solution here, I'll give you a more general exercise:

Suppose ~~$f: \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$~~

$f : [0, +\infty) \to [0, +\infty)$ is a function with the following properties:

(a) $f(0) = 0$;

(b) f is strictly increasing;

(c) $f(a+b) \leq f(a) + f(b) \quad \forall\, a, b \in [0, +\infty)$.

Suppose also that (X, d) is a metric space. Then (X, d^f) is also a metric space, where

$d^f : X \times X \to \mathbb{R}_+$ is defined by

$$d^f(x,y) = f(d(x,y)), \quad \forall\, x, y \in X.$$

Solving the exercise above is quite straightforward.

To see how the above applies to your concrete problem consider $f : [0, +\infty) \to [0, +\infty)$; $f(x) = \frac{x}{1+x}$ and check that the conditions (a), (b), (c) are satisfied

Further comment / problem

Find other examples of functions f satisfying conditions (a), (b), (c) above.

Pb 4 (a) We first prove the following lemma:

Lemma: Suppose d and ρ are metrics on X and $\exists M > 0$ so that

$$d(x,y) \leq M \cdot \rho(x,y) \quad \forall x,y \in X. \qquad (*)$$

Then:

(a) $\forall x \in X$, $\forall r > 0$, $B_\rho(x,r) \subseteq B_d(x, M\cdot r)$

where $B_\rho(x,r) = \{y \mid \rho(x,y) < r\}$ and

$B_d(x, Mr) = \{y \mid d(x,y) < M\cdot r\}$

(b) $\mathcal{T}_d \subseteq \mathcal{T}_\rho$ (so ρ induces a finer topology than d if $(*)$ is satisfied)

Proof of Lemma:

(a) $y \in B_\rho(x,r) \Rightarrow \rho(x,y) < r \overset{(*)}{\Longrightarrow} d(x,y) \leq M\cdot\rho(x,y) < Mr \Rightarrow y \in B_d(x,Mr)$

(b) ~~Since the set of d-open balls is a basis for the topology $\mathcal{T}_d$, it is enough to show that every d-open ball is an open set in $\mathcal{T}_\rho$.~~

Let $U \in \mathcal{T}_d$ and let $x \in U$. By definition of the metric topology, $\exists\, \varepsilon > 0$ s.t. $B_d(x,\varepsilon) \subset U$.

Taking $r = \frac{\varepsilon}{M}$ in part (a) we know that $B_\rho(x, \frac{\varepsilon}{M}) \subseteq B_d(x, \frac{\varepsilon}{M} M) \subset U$

Thus $U \in \mathcal{T}_\rho$ so the Lemma is proved.

If $\exists A, B > 0$ so that $A\rho(x,y) \leq d(x,y) \leq B\rho(x,y)$, we apply the Lemma twice – once for $d(x,y) \leq B\cdot\rho(x,y)$ and once for $\rho(x,y) \leq \frac{1}{A} d(x,y)$ – to conclude $\mathcal{T}_d = \mathcal{T}_\rho$, so the metrics are equivalent.

Pb 4(b) Obviously

$$\max\{(x_i-y_i)^2 \mid i=1,\dots,n\} \le \sum_{i=1}^{n}(x_i-y_i)^2 \le n\cdot\max\{(x_i-y_i)^2 \mid i=1,\dots,n\} \quad \text{so}$$

$$\max\{|x_i-y_i| \mid i=1,\dots,n\} \le \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} \le \sqrt{n}\,\max\{|x_i-y_i| \mid i=1,\dots,n\}.$$

Thus if d^{sq} denotes the square metric in $\mathbb{R}^n$ and d^E denotes the Euclidean metric in $\mathbb{R}^n$,

we proved
$$d^{sq}(x,y) \le d^E(x,y) \le \sqrt{n}\, d^{sq}(x,y)$$
so d^E and d^{sq} are strongly equivalent (note that n is <u>fixed</u>)

4(c) Let $d^{t.c.}(x,y) = \sum_{i=1}^{n}|x_i-y_i|$ be the taxi-cab metric on $\mathbb{R}^n$

It is obvious that
$$d^{sq}(x,y) = \max\{|x_i-y_i|\} \le d^{t.c}(x,y) = \sum_{i=1}^{n}|x_i-y_i| \le n\cdot\max_{i=1,\dots,n}\{|x_i-y_i|\} = n\cdot d^{sq}(x,y)$$
so $d^{t.c}$ and d^{sq} are strongly equivalent

Using the inequalities in 4(b) we also get
$$\frac{1}{\sqrt{n}}d^E(x,y) \le d^{sq}(x,y) \le d^{t.c}(x,y) \le n d^{sq}(x,y) \le n d^E(x,y)$$
so d^{tc} and d^E are strongly equivalent.

By parts (a), (b), (c) we conclude that the Euclidean metric, the square metric and the taxi-cab metric all induce the same topology on $\mathbb{R}^n$

Pb. 4(d) We prove two lemmas

<u>Lemma 1</u>: If (X,d) is a metric space and

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$$ is the metric given in Pb. 3,

then d and d' are equivalent (i.e. they induce the same topology on X)

<u>Lemma 2</u>: If (X,d) is an unbounded metric space then d and d' (from above) are not strongly equivalent.

<u>Definition</u>: (X,d) is an unbounded metric space if $\forall K>0$, $\exists x,y \in X$ with $d(x,y) > K$.

<u>Proof of Lemma 2</u>: Suppose that d and d' are strongly equivalent. Then $\exists A>0$, $\exists B>0$ so that

$$A\,d(x,y) \leq d'(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq B\,d(x,y) \qquad \forall x,y \in X.$$

The right inequality is always true when $B \geq 1$ so it does not lead to any contradiction. But the left inequality does, as we get

$$A\cdot(1+d(x,y)) \leq 1 \qquad \forall x,y \in X$$

But by ~~unboundedness~~ the assumption $\forall K>0, \exists x,y \in X$ with $d(x,y)>K$. Thus $A\cdot x \leq 1$ $\forall x>0$ so $A=0$

Contradiction.

Pb. 4(d) continuation

Proof of Lemma 1:

Note that if $0<\varepsilon<1$ then

$$d'(x,y)<\varepsilon \iff \frac{d(x,y)}{1+d(x,y)}<\varepsilon \iff d(x,y)<\frac{\varepsilon}{1-\varepsilon}$$

Thus $B_{d'}(x,\varepsilon) = B_d(x,\frac{\varepsilon}{1-\varepsilon})$ ~~(taking r = $\frac{\varepsilon}{1-\varepsilon}$)~~

Equivalently, for any $r>0$, $B_d(x,r) = B_{d'}(x,\frac{r}{1+r})$.

Suppose $U\in \mathcal{T}_d$ i.e. U is open w.r.t. metric d and let $x\in U$.
Then $\exists\, r>0$, s.t. $x\in B_d(x,r)\subseteq U$. By above remark

$$x\in B_d(x,r) = B_{d'}(x,\tfrac{r}{1+r})\subseteq U,$$ so U is open w.r.t. d'

Thus $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$.

Similarly if $U\in\mathcal{T}_{d'}$, if $x\in U$, $\exists\, 0<\varepsilon<1$ s.t.

$x\in B_{d'}(x,\varepsilon)\subseteq U$, but this means

$$x\in B_{d'}(x,\varepsilon) = B_d(x,\tfrac{\varepsilon}{1-\varepsilon})\subseteq U$$

so $U\in\mathcal{T}_d$.

Thus $\mathcal{T}_d = \mathcal{T}_{d'}$.

By Lemma 1, Lemma 2, Pb. 4d is solved by considering $\mathbb{R}$ with d the Euclidean metric and d' the metric of Pb 3.
d and d' are equivalent but not strongly equivalent