Homework 3 Topology - Solution Key

<u>Problem 1</u>: We'll show that

$$A^c = \{x \in X \mid f(x) \neq g(x)\} \text{ is open.}$$

Let $x \in A^c$. Then $f(x) \neq g(x)$ and as Y is Hausdorff there exist open sets V_1, V_2 with $f(x) \in V_1$, $g(x) \in V_2$ and $V_1 \cap V_2 = \emptyset$.

As f and g are continuous, $f^{-1}(V_1)$ and $g^{-1}(V_2)$ are open in X. ~~and~~ Also, $x \in f^{-1}(V_1) \cap g^{-1}(V_2)$, since $f(x) \in V_1$ and $g(x) \in V_2$.

We claim that $f^{-1}(V_1) \cap g^{-1}(V_2) \subseteq A^c$.

Indeed if $\tilde{x} \in f^{-1}(V_1) \cap g^{-1}(V_2)$ then $f(\tilde{x}) \in V_1$ and $g(\tilde{x}) \in V_2$ but as $V_1 \cap V_2 = \emptyset$, we must have $f(\tilde{x}) \neq g(\tilde{x})$, so $\tilde{x} \in A^c$.
Thus for any point in A^c we can find a neighborhood contained in A^c. Thus A^c is open, so A is closed.

Problem 2: Fix $x_0 \in X$. We'll show that $h(x) = f(x) \cdot g(x)$ is continuous at x_0.

Observe that $|h(x) - h(x_0)| = |f(x)g(x) - f(x_0)g(x_0)| =$

$$= |f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)| =$$

$$= |g(x) \cdot (f(x) - f(x_0)) + f(x_0)(g(x) - g(x_0))| \leq$$

$$\leq |g(x)| \cdot |f(x) - f(x_0)| + |f(x_0)| \cdot |g(x) - g(x_0)|$$

Let now $\varepsilon > 0$. Since f and g are continuous:

- $\exists\, U_1$ neighborhood of x_0 so that $|g(x) - g(x_0)| < 1$, $\forall x \in U_1$
  Note that if $x \in U_1 \Rightarrow |g(x)| \leq |g(x_0)| + |g(x) - g(x_0)| < 1 + |g(x_0)|$. (1)
- $\exists\, U_2$ neighbd. of x_0 so that $|f(x) - f(x_0)| < \dfrac{\varepsilon}{2(1+|g(x_0)|)}$ (2)
- $\exists\, U_3$ nbd. of x_0 so that $|g(x) - g(x_0)| < \dfrac{\varepsilon}{2(1+|f(x_0)|)}$ (3)

Let $U = U_1 \cap U_2 \cap U_3$. This is an open nbd of x_0.

Moreover, if $x \in U$, then

$$|h(x) - h(x_0)| \leq |g(x)| \cdot |f(x) - f(x_0)| + |f(x_0)| \cdot |g(x) - g(x_0)|$$

$$\underset{\text{by (1),(2),(3)}}{<} (1+|g(x_0)|) \cdot \frac{\varepsilon}{2(1+|g(x_0)|)} + |f(x_0)| \cdot \frac{\varepsilon}{2(1+|f(x_0)|)} <$$

$$\underset{\text{since } \frac{|f(x_0)|}{1+|f(x_0)|} \leq 1}{\leq} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus h is continuous at x_0, and since x_0 was arbitrary, h is continuous on X.

Problem 3: Let $A \subseteq X$ be a subset of $(X, \mathcal{T})$ Hausdorff ~~and let~~

Let $\mathcal{T}_A = \{ U \cap A \mid U \in \mathcal{T} \}$ be the induced topology on A.

If $a_1 \neq a_2$ are two points in A, since $A \subseteq X$, $a_1, a_2 \in X$ and $a_1 \neq a_2$.

Because X is Hausdorff, $\exists\, U_1, U_2 \in \mathcal{T}$, with $a_1 \in U_1$, $a_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

But then $U_1 \cap A \in \mathcal{T}_A$, $U_2 \cap A \in \mathcal{T}_A$

$a_1 \in U_1 \cap A$, $a_2 \in U_2 \cap A$ and

$$(U_1 \cap A) \cap (U_2 \cap A) = U_1 \cap U_2 \cap A = \emptyset$$

Thus $(A, \mathcal{T}_A)$ is Hausdorff.

We proved that the Hausdorff property is hereditary.

Problem 4: We need to find an example of a topological space $(X, \mathcal{T})$, a subspace $(A, \mathcal{T}_A)$ and a subset $B \subseteq X$ so that

$$cl_A(A \cap B) \neq A \cap cl_X(B), \text{ where}$$

cl_A denotes the closure in the subspace topology $\mathcal{T}_A$ and cl_X denotes the closure in $\mathcal{T}_X$.

But we proved that $cl_A(A\cap B) = A \cap cl_X(A \cap B)$, so we need to find examples such that

$$A \cap cl_X(A\cap B) \neq A \cap cl_X(B).$$

One example: Let $(X, \mathcal{T}) = (\mathbb{R}, \mathcal{T}_{usual})$

$$A = \mathbb{Q}, \quad B = \mathbb{R} \setminus \mathbb{Q}$$

Since $A \cap B = \emptyset$, obviously $cl_X(A \cap B) = \emptyset$

Thus $A \cap cl_X(A\cap B) = A \cap \emptyset = \emptyset$

The rightside though is

$$A \cap cl_X(B) = \mathbb{Q} \cap \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{Q} \cap \mathbb{R} = \mathbb{Q} \neq \emptyset$$

Note that the comments above show that the inclusion $cl_A(A \cap B) \subseteq A \cap cl_X(B)$ is always true, but the other inclusion is not always true.