Homework 4 - Topology - Solutions

Pb. 1. The topology $\mathcal{U}$ on $Y=\{1,2,3\}$ induced by f is defined by

$$U \in \mathcal{U} \iff f^{-1}(U) \text{ is open in } (\mathbb{R}, \mathcal{T}_{usual})$$

Note that
$$f^{-1}(\{1\}) = (-\infty, 0) \in \mathcal{T}_{usual}$$
$$f^{-1}(\{3\}) = (0, +\infty) \in \mathcal{T}_{usual}$$
$$f^{-1}(\{2\}) = \{0\} \notin \mathcal{T}_{usual}$$

Also
$$f^{-1}(\{1,2\}) = (-\infty, 0] \notin \mathcal{T}_{usual}$$
$$f^{-1}(\{2,3\}) = [0, +\infty) \notin \mathcal{T}_{usual}, \quad f^{-1}(\{1,3\}) = (-\infty,0)\cup(0,+\infty) \in \mathcal{T}_{usual}$$

Thus $\mathcal{U} = \{\emptyset, \{1\}, \{3\}, \{1,3\}, Y\}$. ∎

Pb. 2 (a) In general, if $f: X \to Y$ is any function then the relation $\sim_f$ on X defined by

$$x_1 \sim_f x_2 \overset{def}{\iff} f(x_1) = f(x_2)$$

is an equivalence relation.

It is reflexive since $f(x) = f(x) \quad \forall x \in X$;

symmetric since $\big(f(x_1) = f(x_2) \Rightarrow f(x_2) = f(x_1)\big) \quad \forall x_1, x_2 \in X$

transitive since $\big[\big(f(x_1) = f(x_2) \wedge f(x_2) = f(x_3)\big) \Rightarrow f(x_1) = f(x_3)\big] \quad \forall x_1, x_2, x_3 \in X$

The relation in the problem is induced by the function $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x + y^2$

Pb. 2 (b) We'll show that the map

$$f: \mathbb{R}^2 \to \mathbb{R} \quad , \quad f(x,y) = x + y^2 \quad \text{is a}$$

quotient map and, thus, $\mathbb{R}^2/_{\sim} \overset{\text{homeo}}{\simeq} (\mathbb{R}, \mathcal{T}_{usual})$

It is obvious that f is continuous and onto (since $f(x,0) = x$)

We show that f is also open, which implies that f is a quotient map.

To show f is open is enough to prove that for any open sets U, V in $\mathbb{R}$ $f(U \times V)$ is open in $\mathbb{R}$.

Let $r_0 \in f(U \times V)$. Then $\exists\ x_0 \in U, y_0 \in V$ s.t. $f(x_0, y_0) = x_0 + y_0^2 = r_0$

Since U is open, $\exists\ \varepsilon > 0$ s.t. $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq U$.

We claim that $(r_0 - \varepsilon, r_0 + \varepsilon) \subseteq f(U \times V)$, which shows $f(U \times V)$ open.

Indeed, if $r \in (r_0 - \varepsilon, r_0 + \varepsilon) \Rightarrow -\varepsilon < r - r_0 = r - y_0^2 - x_0 < \varepsilon$

Let $x := r - y_0^2$. By above inequality, $-\varepsilon < x - x_0 < \varepsilon$ so $x \in (x_0 - \varepsilon, x_0 + \varepsilon) \subseteq U$.

Since $f(x, y_0) = x + y_0^2 = r - y_0^2 + y_0^2 = r$ and $(x, y_0) \in U \times V$ $\Rightarrow$

$\Rightarrow r \in f(U \times V)$.

$\blacksquare$

<u>Pb. 3</u> (a) Let $\pi_j : \mathbb{R}^\omega \to \mathbb{R}$ be the projection on the j-th component, i.e. $\pi_j(\{x_n\}_n) := x_j$.

We know that with respect to both the product topology and the box topology, the projections are continuous (but the product topology is the coarsest with this property).

Note that

$$H = \prod_{n \geq 1} [0, \tfrac{1}{n}] = \bigcap_{n=1}^{\infty} \pi_n^{-1}([0, \tfrac{1}{n}]).$$

But for any n, $[0, \frac{1}{n}]$ is closed in $\mathbb{R}$, so $\pi_n^{-1}([0,\frac{1}{n}])$ is closed in $\mathbb{R}^\omega$ with respect to either the box or the product topology.

Since arbitrary intersection of closed sets is still closed, H is closed with respect to either ~~topology~~ box or product topology on $\mathbb{R}^\omega$

(b) <u>Claim 1</u>: W.r.t. product topology on $\mathbb{R}^\omega$, $\mathrm{int}(H) = \emptyset$.

<u>Claim 2</u>: W.r.t. box topology on $\mathbb{R}^\omega$, $\mathrm{int}(H) = \prod_{n \geq 1} (0, \frac{1}{n})$

<u>Proof Claim 1</u>: Suppose $\bar{x} = \{x_n\}_n \in \mathrm{int}(H)$. Then $\exists\ \bar{x} \in U = \prod_{n=1}^{\infty} U_n \subseteq H$ where $U_n \subseteq \mathbb{R}$ open and $U_n = \mathbb{R}$ for all ~~n except finitely many~~ $n \geq n_0$

But as $U_j = \mathbb{R}$ for some j, then $\pi_j(U) \subseteq \pi_j(H)$ so $\mathbb{R} \subseteq [0, \frac{1}{j}]$ $\lightning$.

Thus $\mathrm{int}(H) = \emptyset$ w.r.t. product topology on $\mathbb{R}^\omega$

Pb. 3(b) continuation

Proof of Claim 2:

$$\left.\begin{array}{l}\prod_{n\geq 1}(0,\frac{1}{n}) \in \mathcal{T}_{box} \\ \prod_{n\geq 1}(0,\frac{1}{n}) \subseteq H\end{array}\right\} \Rightarrow \prod_{n\geq 1}(0,\frac{1}{n}) \subseteq \mathrm{int}(H)$$

Let now $\bar{x} = \{x_n\}_n \in \mathrm{int}(H)$

Then there is a set $U = \prod_{n\geq 1} U_n$ of the basis of the box topology (thus all U_n's are open)

so that $\bar{x} \in U \subseteq \mathrm{int}(H) \subseteq H$.

By projecting on the n^{th} component the inclusion above

we have $x_n = \pi_n(\bar{x}) \in \pi_n(U) = U_n \subseteq \pi_n(H) = [0,\frac{1}{n}]$

Since U_n is open $\Rightarrow U_n \subseteq \mathrm{int}[0,\frac{1}{n}] = (0,\frac{1}{n})$

Thus, $\forall n$, $x_n \in (0,\frac{1}{n})$, so $\bar{x} \in \prod_{n\geq 1}(0,\frac{1}{n})$

We thus also proved $\mathrm{int}(H) \subseteq \prod_{n\geq 1}(0,\frac{1}{n})$

so $\prod_{n\geq 1}(0,\frac{1}{n}) = \mathrm{int}(H)$