Name: Solution Key PantherID: ____________

Midterm - Topology – Fall 2015

1. (20 pts) Define each of the following:

See text or notes for Pb. 1

(i) A metric on a set X

(ii) The set of limit points A' of a set A in a topological space $(X, \mathcal{T})$

(iii) A local basis at a point x in a topological space $(X, \mathcal{T})$

(iv) A second countable topological space

(v) A homeomorphism

2. (6 pts) Give an example of a topological space $(X, \mathcal{T})$, a subspace $(A, \mathcal{T}_A)$ of $(X, \mathcal{T})$, and a closed set in $(A, \mathcal{T}_A)$ that is not closed in $(X, \mathcal{T})$.

Many possible examples.

One of them: $(X, \mathcal{T}) = (\mathbb{R}, \mathcal{T}_{usual})$

$A = (0,1)$, $\mathcal{T}_A$ = the subspace topology

$B = (0, \frac{1}{2}]$ is closed in A (since $B = A \cap [0, \frac{1}{2}]$ with $[0, \frac{1}{2}]$ closed in $\mathbb{R}$)

but $B = (0, \frac{1}{2}]$ is not closed in $\mathbb{R}$ (since $\overline{B} = [0, \frac{1}{2}]$)

(With $A = (0,1)$, you could even take $B = (0,1) = A$)

3. (12 pts) Let $(X, \mathcal{T})$ be a topological space and let $f, g : X \to \mathbb{R}$ be continuous functions. Show that the function $h : X \to \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ is continuous.

Fix a point $x_0 \in X$. Let $\varepsilon > 0$. Since f is continuous at x_0,

$\exists\ U_1$ nbd of x_0 s.t. $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$ $\forall x \in U_1$

Since g is continuous at x_0, $\exists\ U_2$ nbd. of x_0 s.t.

$$|g(x) - g(x_0)| < \frac{\varepsilon}{2}$$

Then $\forall x \in U_1 \cap U_2 \Rightarrow |h(x) - h(x_0)| = |f(x) + g(x) - f(x_0) - g(x_0)| \leq$

$$\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since x_0 was arbitrary, it follows that $h(x)$ is continuous at all points in X.

4. (20 pts) Let $X = [0, +\infty)$ be the set of non-negative real numbers and let $\mathcal{T} \subseteq \mathcal{P}(X)$ be defined by

$$\mathcal{T} := \{U \in \mathcal{P}(X) \mid U = \emptyset, \text{ or } U = X, \text{ or } U = (a, +\infty), \text{ for some } a > 0\}$$

(a) (8 pts) Show that $\mathcal{T}$ is a topology on X.

(b) (6 pts) Consider the sequence $\{x_n\}_{n=1}^{\infty}$ in X, where $x_n = n$, for every positive integer n. Is the sequence $\{x_n\}_{n=1}^{\infty}$ convergent in $(X, \mathcal{T})$? Justify your answer.

(c) (6 pts) Is $(X, \mathcal{T})$ a metrizable topological space? Justify your answer.

(a) $\emptyset, X \in \mathcal{T}$ by definition of $\mathcal{T}$

If $U_1, \dots, U_n \in \mathcal{T}$ need to show that $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$

w.l.o.g assume $U_i = (a_i, +\infty)$ for $a_i \geq 0$.

Then $\bigcap_{i=1}^{n} = (\max_{i=1,n} a_i, +\infty) \in \mathcal{T}$.

If $\{U_j\}_{j \in J}$ is a collection with $U_j \in \mathcal{T}$ $\forall j \in J$ we need to show $\bigcup_{j \in J} U_j \in \mathcal{T}$

w.l.o.g. assume $U_j = (a_j, +\infty)$ for $a_j \geq 0$ $\forall j \in J$.

Since the set $\{a_j \mid j \in J\}$ is a set of real numbers bounded from below (by 0), the infimum of this set is a real number.

Then $\bigcup_{j \in J} U_j = (\inf\{a_j \mid j \in J\}, +\infty)$, so $\bigcup_{j \in J} U_j \in \mathcal{T}$
claim

Thus $\mathcal{T}$ is a topology (modulo the claim)

Pb. 4 continuation

Proof of claim for part (a). Let $\underline{a = \inf\{a_j \mid j \in J\}}$.

"$\subseteq$" Let $x \in \bigcup_{j \in J} U_j \Rightarrow \exists\, j_0 \in J$ s.t. $x \in U_{j_0} = (a_{j_0}, +\infty)$

$\Rightarrow x > a_{j_0} \Rightarrow x > \inf\{a_j \mid j \in J\}$

$\Rightarrow x \in (a, +\infty)$

"$\supseteq$" Let $x \in (a, +\infty) \Rightarrow x > a = \inf\{a_j \mid j \in J\} \Rightarrow$

$\Rightarrow \exists\, j_0 \in J$ s.t. $x > a_{j_0} \Rightarrow$

$\Rightarrow x \in U_{j_0} = (a_{j_0}, +\infty) \Rightarrow x \in \bigcup_{j \in J} U_j$

(b) We claim that $\{n\}_{n=1}^{\infty}$ converges to *any* $y \in X$

Indeed if $y = 0$, then the only nbd of 0 is X itself and $n \in X$ $\forall n \geq 1$.

If ~~$y > a > 0$~~ $y > 0$, then the nbds of y are (a, ∞) with $0 \leq a < y$. But $(a, +\infty)$ contains all except finitely many n, $n \in \mathbb{N}$ (since $\exists\, n_0 \in \mathbb{N}$ s.t. $n > a$, $\forall n \geq n_0$)
↑ Archimedean property

(c) The space $(X, \mathcal{T})$ is not Hausdorff as any two open sets will have non-empty intersection.

Thus $(X, \mathcal{T})$ is not metrizable, since any metric space is Hausdorff

(Also, $(X, \mathcal{T})$ is not Hausdorff by the observation in (b), as limits of sequences are not unique in $(X, \mathcal{T})$)

5. (14 pts) Let $A = (-1, 1) = \{x \in \mathbb{R} \mid -1 < x < 1\}$. Find the closure, the interior and boundary of A in $\mathbb{R}$ with the lower limit topology. Justify your answers.

~~Claim~~ a) $\text{int}_{\mathcal{T}_{\ell.\ell.}}(A) = (-1,1)$ since $(-1,1)$ is open in the lower limit topology as well (we know $\mathcal{T}_{usual} \subseteq \mathcal{T}_{\ell.\ell.}$)

Claim (b) $cl_{\mathcal{T}_{\ell.\ell.}}(A) = [-1,1)$

Proof of (b): $[-1,1)$ is closed in $\mathcal{T}_{\ell.\ell.}$ since $\mathbb{R} \setminus [-1,1) = (-\infty,-1) \cup [1,+\infty) \in \mathcal{T}_{\ell.\ell.}$

But $A \subseteq [-1,1)$ so $cl(A) \subseteq [-1,1)$

Also $A = (-1,1) \subseteq cl(A)$ and $-1 \in cl(A)$ since any nbd of -1 contains an interval of the form $[-1,a)$ with $a > -1$, ~~so~~ ~~then~~ and $[-1,a) \cap (-1,1) \neq \emptyset$.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$ $(-1,a) \neq \emptyset$

(c) $bd_{\mathcal{T}_{\ell.\ell.}}(A) = cl_{\mathcal{T}_{\ell.\ell.}}(A) - \text{int}_{\mathcal{T}_{\ell.\ell.}}(A) = \underline{\underline{\{-1\}}}$

See also Exp. 11 section 2.3. text

6. (12 pts) Consider $\mathbb{R}^\omega$, the countable infinite product of $\mathbb{R}$ with itself.
Define the function $f : \mathbb{R} \to \mathbb{R}^\omega$, by $f(t) = (t, t, t, \ldots)$.
Show that if $\mathbb{R}^\omega$ is given the box topology then f is not continuous.

This example was presented during lecture in class

It is enough to show that f is not continuous at $t=0$.

$f(0) = (0,0,0,\ldots)$

Let ~~U~~ $U = \prod_{n=1}^{\infty} U_n$ where $U_n = (-\frac{1}{n}, \frac{1}{n}) \subseteq \mathbb{R}$.

U is an open nbd of $(0,0,0,\ldots)$ in the box topology on $\mathbb{R}^\omega$.

$t \in f^{-1}(U) \iff (t,t,t,\ldots) \in \prod_{n=1}^{\infty} U_n \iff -\frac{1}{n} < t < \frac{1}{n} \quad \forall$ ~~n~~ $n \geq 1$, nat

$\Updownarrow$

$\underline{\underline{t=0}}$

~~Thus~~

Thus $f^{-1}(U) = \{0\}$ but $\{0\}$ is not open in $(\mathbb{R}, \mathcal{T}_{usual})$

Thus f is not continuous

Choose TWO of the following THREE problems on this page.

7. (12 pts) Let (X, d) be a complete metric space. Show that a subspace A is complete (with the induced metric) if and only if A is closed in X.

8. (12 pts) Let A be an open subset of a separable space $(X, \mathcal{T})$. Prove that $(A, \mathcal{T}_A)$ is separable.

9. (12 pts) Let $(X, \mathcal{T})$ be topological a space and let $\mathcal{U}$ denote the product topology on $X \times X$. Define the diagonal $\Delta \subset X \times X$ by $\Delta = \{(x, x) \mid x \in X\}$. Show that $(X, \mathcal{T})$ is Hausdorff if and only if Δ is a closed subset in $(X \times X, \mathcal{U})$.

"$\Rightarrow$"

$\boxed{7.}$ Suppose (A, d_A) is complete. We want to show $\bar{A} = A$, where the closure is taken in X.

Let $x \in \bar{A}$. Since (X, d) is a metric space (so 1st countable), it follows that there is a sequence $\{x_n\}_n$ with $x_n \in A$, $\forall n$ so that $\{x_n\}_n$ converges to x (in X). Since $\{x_n\}_n$ converges in X, then $\{x_n\}_n$ is Cauchy (in X), but then $\{x_n\}_n$ is also Cauchy in A since $x_n \in A$, $\forall n$ and $d(x_n, x_m) = d_A(x_n, x_m)$ $\forall n, m$.

As (A, d_A) is complete, by assumption, $\{x_n\}_n$ is convergent to a point in A. Since a metric space is Hausdorff, ~~a sequence~~ the limit of a convergent sequence is unique, hence $x \in A$.

"$\Leftarrow$" Suppose now $\bar{A} = A$ and we want to show that (A, d_A) is complete.

Let $\{x_n\}_n$ a Cauchy sequence in A. Then $\{x_n\}_n$ is a Cauchy sequence in X, and since (X, d) is complete it follows that $\{x_n\}_n$ is convergent to a point $x \in X$ (again limit is unique)

But since $x_n \in A$, $\forall n$ and $x_n \to x$ $\Bigg\} \Rightarrow x \in \bar{A} \underset{\substack{\uparrow \\ \text{by assumption}}}{=} A$

Thus any Cauchy sequence in A converges to a point in A thus (A, d_A) is complete

Note: I think this is also proved in the text, but even if you did not read this before, it's the type of proof that you should be able to do.

Pb. 8. $(X, \mathcal{T})$ separable $\Rightarrow \exists\, Q \subseteq X$, Q countable and dense in X.

Fix such a Q and consider $Q \cap A$.

$Q \cap A \subseteq Q$ so $Q \cap A$ is countable (subset of a countable set)

We show that the closure of $Q \cap A$ in A is A, i.e.

$$cl_A(Q \cap A) = A$$

"$\subseteq$" is obvious (as A is the "universe" now)

"$\supseteq$" Let $a \in A$ and let $a \in U_A$ be a nbd. in A of a.

By definition of subspace topology, $U_A = U \cap A$ for some $U \in \mathcal{T}$.

~~Since $a \in U_A = U \cap A$, we have $a \in U$~~

~~Since $\overline{Q} = X$~~ $\Rightarrow$

Since A is open in X, it follows that and $a \in X$ $U_A = U \cap A$ is open in X

Thus $U \cap A$ is an open nbd. of a in X
Since $\overline{Q} = X$ and $a \in X$ $\Big\} \Rightarrow (U \cap A) \cap Q \neq \emptyset$

But $\emptyset \neq (U \cap A) \cap Q = (U \cap A) \cap (A \cap Q) = U_A \cap (Q \cap A)$

so we proved that $a \in cl_A(Q \cap A)$

Thus $A \subseteq cl_A(Q \cap A)$ so $A = cl_A(Q \cap A)$

Hence we showed that $(A, \mathcal{T}_A)$ is separable.

Pb.9

$\Delta = \{(x,x) \mid x \in X\}$ is closed in $X \times X \iff$

$\iff (X \times X) \setminus \Delta$ is open in $X \times X \iff$

$\iff$ (*) $\forall (x,y) \in (X \times X) \setminus \Delta$, $\exists U_1 \times U_2$ element of the basis of product topology on $X \times X$

so that $(x,y) \in U_1 \times U_2 \subseteq (X \times X) \setminus \Delta$

Note that $(x,y) \in (X \times X) \setminus \Delta \iff x, y \in X$ and $x \neq y$.

Thus $U_1 \times U_2 \subseteq (X \times X) \setminus \Delta \iff \forall u_1 \in U_1, \forall u_2 \in U_2, u_1 \neq u_2 \iff$

$\iff U_1 \cap U_2 = \emptyset$

Thus statement (*) above is equivalent to

$\forall x, y \in X$ with $x \neq y$, $\exists U_1$ nbd of x, $\exists U_2$ nbd of y such that $U_1 \cap U_2 = \emptyset$

But this is exactly the definition of X being Hausdorff.

Thus Δ is closed in $X \times X \iff X$ is Hausdorff.