Exam 1 Topology - Spring 2008 Name: ____________________

1. (20 pts) Define each of the following:

(i) topology on a set X

(ii) closure of a set $A \subset X$, where X is a topological space

(iii) the subspace topology on A, where $A \subset X$ and X is a topological space

(iv) homeomorphism

2. (18 pts) Let $f : X \to Y$ be a function and let A, B subsets of X.

(i) (10 pts) Show that $f(A \bigcap B) \subseteq f(A) \bigcap f(B)$, and give an example when the inclusion is strict.

(ii) (8 pts) Show that if f is injective, equality holds in (i).

3. (20 pts) (a) (10 pts) If d_1 is a metric on X and d_2 is a metric on Y, show that

$$d((x, y), (x', y')) = d_1(x, x') + d_2(y, y') \text{ is a metric on } X \times Y.$$

(b) (10 pts) Show that the topology induced by d is the same as the product topology on $X \times Y$.

4. (16 pts) Consider the product, uniform and box topologies on $\mathbf{R}^\omega$. Consider the sequence

$$\mathbf{x}_1 = (1, 1, 1, 1, ...),\ \mathbf{x}_2 = (0, 1/2, 1/2, 1/2, ...),\ \mathbf{x}_3 = (0, 0, 1/3, 1/3, ...),\ ...$$

In which topologies (if any), does the following sequence converge? Justify your answer.

5. (16 pts) Let X, Y be topological spaces and let $f, g : X \to Y$ be continuous functions. Assume that Y is Hausdorff. Show that $\{x \,|\, f(x) = g(x)\}$ is closed in X.

6. (20 pts) (a) (10 pts) Show that the collection $\mathcal{C} = \{[a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$ is a basis for a topology on $\mathbf{R}$.

(b) (10 pts) Denote by $\mathcal{T}'$ the topology generated by the collection $\mathcal{C}$ in part (a), by $\mathcal{T}$ the standard topology and by $\mathcal{T}_l$ the lower limit topology on $\mathbf{R}$. Show that $\mathcal{T} \subset \mathcal{T}' \subset \mathcal{T}_{\updownarrow}$, with both inclusions being strict.