Name: ____________________ **SSN:** ____________________

Exam 1 MAA 4211 Spring 2002

To receive credit you MUST show your work.

1. (15 pts) Prove that

$$(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2), \text{ for all } a_1, a_2, b_1, b_2 \in \mathbf{R}.$$

When does equality hold?

Solution: Expanding both sides, the inequality is equivalent to:

$$a_1^2b_1^2 + a_2^2b_2^2 + 2a_1a_2b_1b_2 \leq a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2 \,,$$

or further to

$$0 \leq a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2.$$

But in the right hand side we recognize that we have a perfect square, thus the given inequality is equivalent to:

$$0 \leq (a_1b_2 - a_2b_1)^2,$$

which is obviously true.

Equality holds if and only if $a_1b_2 - a_2b_1 = 0$. □

Observation 1: Note that, assuming that b_1, b_2 are non-zero, the condition $a_1b_2 - a_2b_1 = 0$ can also be written as $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. Thus one can say that equality holds when the terms b_i are proportional to the terms a_i.

Observation 2: The inequality in this problem is a particular case of the so called *Cauchy-Schwarz inequality* which states that:

$$(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2), \text{ for all } a_1, ...a_n, b_1, ...b_n \in \mathbf{R}, n \in \mathbf{N}.$$

The Cauchy-Schwarz inequality is of fundamental importance in many areas of mathematics. Try to also prove the general case by expanding both sides and grouping the terms to form some perfect squares.

2. (15 pts) (a) (5 pts) State the Bolzano-Weierstrass theorem.

Solution: Any bounded sequence of real numbers has a convergent subsequence.

(b) (10 pts) Is the converse of the Bolzano-Weierstrass theorem true? Justify your answer.

Solution: The converse of the Bolzano-Weierstrass theorem would be: If a sequence has a convergent subsequence, then the sequence is bounded.

This statement is false. One example to show this would be for instance $x_n = n + (-1)^n n$. We have that $x_{2k} = 2k$, so the sequence is not bounded, as we can find terms of the sequence larger than any constant M (by taking $k \in \mathbf{N}$ large enough so that $2k > M$). But on the other hand, $x_{2k+1} = 0$, for any $k \in \mathbf{N}$, so the subsequence $\{x_{2k+1}\}_k$ trivially converges to 0. □

Observation: Of course many other examples like this could be constructed and you should come up with your own.

3. (20 pts) Let f be a real function. Write the definition for each of the following:

$$\lim_{x\to a-} f(x) = L, \text{ where } a, L \in \mathbf{R}.$$

Solution:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, (a-\delta < x < a \ \rightarrow \ |f(x) - L| < \epsilon). \ \square$$

$$\lim_{x\to +\infty} f(x) = -\infty.$$

Solution:

$$\forall m \in \mathbf{R}, \exists M \in \mathbf{R} \text{ such that } \forall x, (x > M \ \rightarrow \ f(x) < m). \ \square$$

4. (20 pts) (a) (5 pts) Prove that if x is an upper bound for a set $E \subset \mathbf{R}$ and $x \in E$, then x is the supremum of E.

Solution: It is given that x is an upper bound for E. Suppose M is another upper bound for E. Since $x \in E$, it follows that $x \leq M$. Thus x is the smallest upper bound of E, thus, it is the supremum of E. $\square$

(b) (5 pts) State without proof an analogous statement for the infimum of E.

Solution: If x is a lower bound for a set $E \subset \mathbf{R}$ and $x \in E$, then x is the infimum of the set E. $\square$

(c) (10 pts) Find the supremum and the infimum of the set $E = \{1 + (-1)^n(1 + \frac{1}{n}) \mid n \in \mathbf{N}\}$.

Solution: The set E can be alternatively described as $E = \{-\frac{1}{2k-1}, 2 + \frac{1}{2k} \mid k \in \mathbf{N}\}$. The following inequalities are obvious

$$-1 \leq -\frac{1}{2k-1} < 2 + \frac{1}{2k} \leq \frac{5}{2}, \ \forall k \in \mathbf{N}.$$

Thus the set E is bounded from below by -1 and bounded from above by $5/2$. But -1 and $5/2$ are elements of the set E. Thus from parts (a) and (b), it follows that $\sup E = 5/2$, $\inf E = -1$. $\square$

5. (20 pts) Suppose $2 \leq x_1 < 3$ and $x_{n+1} = 2 + \sqrt{x_n - 2}$ for $n \in \mathbf{N}$. Study the monotonicity and the convergence of the sequence $\{x_n\}_n$. Completely justify all your claims.

Solution: First we prove by induction that $2 \leq x_n < 3$ for any $n \in \mathbf{N}$.
For $n = 1$ this is given by hypothesis. Assume that $2 \leq x_n < 3$ for a given n. Then $0 \leq x_n - 2 < 1$, thus $0 \leq \sqrt{x_n - 2} < 1$, so $2 \leq x_{n+1} = 2 + \sqrt{x_n - 2} < 3$. Thus $2 \leq x_n < 3$ for any $n \in \mathbf{N}$, so we proved that the sequence is bounded.
Next we show that the sequence is increasing. This can be done again by induction, but it is also possible to do it directly, using the bounds we obtained in the first step.
Recall that if $y \in \mathbf{R}$, $0 \leq y < 1$, we have $\sqrt{y} \geq y$. Since we showed $2 \leq x_n < 3$, we thus have $0 \leq x_n - 2 < 1$, so $\sqrt{x_n - 2} \geq x_n - 2$. This implies that $x_{n+1} = 2 + \sqrt{x_n - 2} \geq x_n$, for an arbitrary $n \in \mathbf{N}$, thus our sequence is increasing. Note also that if $x_1 = 2$, then $x_n = 2$ for any n (can be shown immediately by induction), so the sequence is constant, hence trivially convergent to 2. If $x_1 > 2$, then all inequalities are strict, so our sequence is strictly increasing.
We showed that $\{x_n\}_n$ is bounded and increasing, so by the monotone convergence theorem, the sequence is convergent. Let's denote the limit with L. Taking the limit as $n \to \infty$ in the recursive relation, we get $L = 2 + \sqrt{L - 2}$. Solving we get two solutions $L_1 = 2$ and $L_2 = 3$. As we noted above, if $x_1 = 2$, the sequence is constant 2, so the limit is 2. If $x_1 > 2$, then $2 < x_1 < x_2 < ... < x_n < ... < 3$, so the limit is the supremum of the sequence, and the supremum cannot be 2. Thus the limit is 3. □

Observation: Another solution, perhaps even shorter, can be obtained by showing from the recursive relation that $x_n = 2 + (x_1 - 2)^{1/(2n)}$, $\forall n \in \mathbf{N}$. Try to show this and then get the rest of the problem based on this observation (look also at the Example 2.21, page 45 textbook).

6. (20 pts) Show that a sequence $\{x_n\}_n$ is not bounded from above if and only if there exists a subsequence $\{x_{n_k}\}_k$ such that $x_{n_k} \to \infty$ as $k \to \infty$. (Note that you have to prove both implications.)

Solution: First of all, $\{x_n\}_n$ is bounded from above, by definition, if and only if there exists a constant $M \in \mathbf{R}$, such that $x_n \leq M$, $\forall n \in \mathbf{N}$. Negating this, $\{x_n\}_n$ is not bounded from above if and only if

$$\forall M \in \mathbf{R},\ \exists n_M \in \mathbf{N},\ \text{such that } x_{n_M} > M. \tag{1}$$

In the above the subscript M for n_M, just indicates the dependence on M of the rank.

Now let us prove the equivalence asked in the statement.

($\Leftarrow$) This implication is easy. Assume that a subsequence $\{x_{n_k}\}_k \to \infty$ as $k \to \infty$. By definition, $\forall M \in \mathbf{R}$, $\exists K \in \mathbf{N}$, such that if $k \geq K$, then $x_{n_k} > M$. Thus relation (1) is trivially satisfied, taking n_M to be, for instance n_K.

($\Rightarrow$) This is a bit harder. We assume that relation (1) is true and we'll construct inductively a subsequence $\{x_{n_k}\}_k$ which will approach ∞ as $k \to \infty$.

To pick the first term of our subsequence, apply first (1) with $M = M_1 = 1$. It follows that there exists $n_1 \in \mathbf{N}$ such that $x_{n_1} > 1$. Now the idea is to apply (1) with a bigger and bigger M. But just choosing $M = 2, M = 3$, etc. may not work, because we have no guarantee that the ranks n_2, n_3, etc. which we get from (1) will be in increasing order.

Here is how we pick the second term of the subsequence. Let $M_2 = \max\{2, x_1, x_2, ..., x_{n_1}\}$ and apply (1) with $M = M_2$. It follows that there exists $n_2 \in \mathbf{N}$ such that $x_{n_2} > M_2$. From the choice of M_2 we deduce two things. First, $x_{n_2} > M_2 \geq 2$. Secondly, $x_{n_2} > M_2 \geq x_l$, for any $l \in \{1, 2, ..., n_1\}$; in particular, we get that $n_2 > n_1$ because otherwise x_{n_2} would be equal to one of the x_l, $l \in \{1, 2, ..., n_1\}$.

Suppose now that we picked $n_1 < n_2 < ... < n_k$, such that $x_{n_l} > l$, for any $l \in \{1, 2, ..., k\}$ and we'll construct the $k+1$-th term of the subsequence. Let $M_{k+1} = \max\{2, x_1, x_2, ..., x_{n_k}\}$ and apply (1) with $M = M_{k+1}$. It follows that there exists $n_{k+1} \in \mathbf{N}$ such that $x_{n_{k+1}} > M_{k+1}$. From the choice of M_{k+1} it follows as above that $n_{k+1} > n_k$ and that $x_{n_{k+1}} > k + 1$.

Thus by induction we construct the subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$, with the property that $x_{n_k} > k$, for any $k \in \mathbf{N}$. The (extended) comparison Theorem implies immediately that $x_{n_k} \to \infty$ as $k \to \infty$. $\square$