Name: ______________________________ **SSN:** ______________________

Exam 2 MAA 4211 Spring 2002

To receive credit you MUST show your work.

1. (15 pts) State the Mean Value Theorem.

Solution: If f is continuous on $[a, b]$ and is differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. $\square$

2. (20 pts) Suppose that

$$f_\alpha(x) = \begin{cases} |x|^\alpha \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) (10 pts) If $\alpha > 0$, show that f is continuous at $x = 0$.

Solution: If $\alpha > 0$, $|x|^\alpha \to 0$ as $x \to 0$. Since $\cos\frac{1}{x}$ is bounded on its domain of definition, by the squeeze theorem for functions it follows that

$$\lim_{x\to 0} f_\alpha(x) = \lim_{x\to 0} |x|^\alpha \cos\frac{1}{x} = 0 = f_\alpha(0). \ \square$$

(b) (10 pts) If $\alpha > 1$, show that f is differentiable at $x = 0$.

Solution: By definition, we need to show that the limit

$$\lim_{x\to 0} \frac{f_\alpha(x) - f_\alpha(0)}{x}$$

exists and is finite. But

$$\lim_{x\to 0} \frac{f_\alpha(x) - f_\alpha(0)}{x} = \lim_{x\to 0} \frac{|x|^\alpha}{x} \cos\frac{1}{x}.$$

As in part (a), the limit is 0, because $\frac{|x|^\alpha}{x}$ approaches 0 as x approaches 0 (because $\alpha > 1$ now) and $\cos\frac{1}{x}$ is bounded. Thus the derivative at $x = 0$ exists and is equal to 0. $\square$

3. (25 pts) (a) (15 pts) Suppose that f is differentiable on a nonempty, open interval (a, b), with f' bounded on (a, b). Prove that f is uniformly continuous on (a, b).

Solution: By assumption $\exists\, M > 0$ such that $|f'(x)| \leq M$, for any $x \in (a, b)$.
Let $\epsilon > 0$ and take $\delta = \epsilon/M$. Let $y, z \in (a, b)$ such that $|z - y| < \delta$. By the mean value theorem applied to f on the interval between y and z, there exists c such that $f(z) - f(y) = f'(c)(z - y)$. Thus

$$|f(z) - f(y)| = |f'(c)||z - y| \leq M|z - y| < M\delta = \epsilon.$$

Since this is true for any y, z such that $|z - y| < \delta$, it follows that f is uniformly continuous on (a, b). □

(b) (10 pts) Give an example to show that if the hypothesis f' bounded on (a, b) is omitted, then the statement of (a) is no longer true.

Solution: Let $f : (0, 1) \to \mathbf{R}$, defined by $f(x) = \ln x$. f is differentiable on $(0, 1)$, and its derivative $f'(x) = 1/x$ is clearly not bounded on $(0, 1)$. The function f is not uniformly continuous because it cannot be extended by continuity at 0 (the limit of f as $x \to 0_+$ is $-\infty$). □

4. (15 pts) Show that $\ln(x + 1) \leq x$, for all $x \geq 0$.

Solution: Let $f(x) = x - \ln(x + 1)$. Note that for any $x \geq 0$, $f'(x) = 1 - 1/(x + 1) \geq 0$. Thus f is increasing on the interval $[0, \infty)$. Thus if $x \geq 0$, then $f(x) \geq f(0)$. Noting that $f(0) = 0$, this is exactly the inequality we had to prove. □

5. (15 pts) Suppose $[a, b]$ is a closed, bounded, nondegenerate interval. Is the following statement true? *For any continuous function $f : [a, b] \to \mathbf{R}$, the function $|f|$, defined by $|f|(x) = |f(x)|$, is integrable on $[a, b]$.* Briefly justify your answer.

Solution: Yes, since f is continuous on $[a, b]$, so is $|f|$, hence, by theorem 5.10, $|f|$ is integrable on $[a, b]$. $\square$

6. (20 pts) Suppose $f : [a, b] \to \mathbf{R}$ is continuous and increasing. Prove that $\sup f(E) = f(\sup E)$ for every nonempty set $E \subseteq [a, b]$.

Solution: Let $E \subseteq [a, b]$. Thus E is bounded, so $s = \sup E$ is finite and $s \in [a, b]$. Because f is increasing, for any $x \in [a, b]$, we have $f(a) \leq f(x) \leq f(b)$. In particular $f(E)$ is bounded, so $\sup f(E)$ is finite.

Again because f is increasing and $s \geq x$, $\forall x \in E$, we have $f(s) \geq f(x)$ $\forall x \in E$. Thus, $f(s)$ is an upper bound for $f(E)$, so we get $f(\text{sup}E) \geq \sup f(E)$.

To obtain the other inequality, let x_n be a sequence of elements from E such that $x_n \to \text{sup}E$ (such a sequence exists, by the approximation property of the supremum). Then $\sup f(E) \geq f(x_n)$, for any $n \in \mathbf{N}$ (because $x_n \in E$). Taking the limit on n in this inequality and using the fact that f is continuous, we get $\sup f(E) \geq f(s) = f(\text{sup}E)$. $\square$