Solution Homework 2: MAA 4211 Spring 2002

4, page 23. Given $a \in \mathbf{R}$ and $n \in \mathbf{N}$, apply the density theorem of rational numbers to $a - 1/n < a + 1/n$ (the inequality holds because $n > 0$). It follows that there exists $r_n \in \mathbf{Q}$ such that $a - 1/n < r_n < a + 1/n$. This double inequality is equivalent to $|a - r_n| < 1/n$. $\square$

8, page 24. Let E_n be the set $\{x_n, x_{n+1}, ...\}$.

Part 1: Since $|x_n| < M$, $\forall n \in \mathbf{N}$, it follows that M is an upper bound for the set E_n. By the completeness axiom, E_n has a (finite) real number supremum denoted by s_n. This is true for any n, so let s_{n+1} be the supremum of E_{n+1}. We next show that $s_n \geq s_{n+1}$. From the definition of the supremum, s_n is an upper bound for E_n, thus $s_n \geq x_i$, for all $i \in \{n, n+1, n+2, ...\}$. In particular, it follows that s_n is an upper bound for the set E_{n+1} as well. But, again by the definition of the supremum, s_{n+1} is the *lowest* upper bound of E_{n+1}. Thus $s_n \geq s_{n+1}$ and since this is true for any n, we proved the statement regarding the suprema.

Part 2: Since $|x_n| < M$, $\forall n \in \mathbf{N}$, it also follows that M is an upper bound for the set $-E_n$. By the completeness axiom, the set $-E_n$ has a (finite) real number supremum which we denote by u_n. From Theorem 1.28, it then follows that the set $E_n = -(-E_n)$ has a real infimum t_n and $t_n = -u_n$. By applying Part 1 to the set $-E_n$, we have $u_1 \geq u_2 \geq ... \geq u_n \geq u_{n+1}...$ and multiplying this by -1, we get $t_1 \leq t_2 \leq ... \leq t_n \leq t_{n+1}....$ $\square$