Solution Homework 4: MAA 4211 Spring 2002

9, page 47. (a) We prove by induction that $0 < y_n < x_n$, $\forall n \in \mathbf{N}$. The statement for $n = 1$ is true, by hypothesis. Now assume $0 < y_n < x_n$. From the arithmetic/geometric mean inequality of positive numbers we have

$$y_{n+1} = \sqrt{x_n y_n} < \frac{x_n + y_n}{2} = x_{n+1},$$

with strict inequality because $y_n < x_n$. □

(b) By (a), $y_{n+1} = \sqrt{x_n y_n} > \sqrt{y_n^2} = y_n$, so the sequence $\{y_n\}$ is (strictly) increasing. Also by (a), $x_{n+1} = \frac{x_n+y_n}{2} < \frac{x_n+x_n}{2} = x_n$, thus $\{x_n\}$ is (strictly) decreasing. We then also have for any $n > 1$, that $y_1 < y_n < x_n < x_1$, so both sequences x_n, y_n are bounded below and above by respectively y_1 and x_1. □

(c) Again use mathematical induction. For $n = 1$, using (a) we have $0 < x_2 - y_2 = \frac{x_1+y_1}{2} - y_2$. Since $y_1 < y_2$ (by (b)), we have further

$$0 < x_2 - y_2 = \frac{x_1 + y_1}{2} - y_2 < \frac{x_1 + y_1}{2} - y_1 = \frac{x_1 - y_1}{2}.$$

Now assume that

$$0 < x_n - y_n < \frac{x_1 - y_1}{2^{n-1}}$$

holds for an arbitrary $n > 1$. Then using again (a) and the fact that $y_n < y_{n+1}$, we have

$$0 < x_{n+1} - y_{n+1} = \frac{x_n + y_n}{2} - y_{n+1} < \frac{x_n + y_n}{2} - y_n = \frac{x_n - y_n}{2} < \frac{x_1 - y_1}{2^n},$$

where for the last inequality we used the inductive assumption. □

(d) By the monotone convergence theorem both sequences x_n and y_n are convergent. On the other hand, using the squeeze theorem in (c), we get $\lim_{n\to\infty}(x_n - y_n) = 0$. Thus using the behavior of the limit of convergent sequences, we get $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. □