Solution Homework 8: MAA 4211 Spring 2002

5, page 92. (a) If f is differentiable at a then f is continuous at a. Since $f(a) \neq 0$, it follows from a result done in Chapter 3 (see lemma 3.28, or Ex. 4, p. 78) that $f(x) \neq 0$, for any x in an interval $I = (a - \delta, a + \delta)$ around a. Thus for any $h \in (-\delta, \delta)$, $f(a + h) \neq 0$. $\square$

(b) We need to show that the limit

$$\lim_{h \to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h} \text{ exists, and is equal to } -\frac{f'(a)}{f^2(a)}.$$

As $f(a) \neq 0$, we have that $\frac{1}{f(a)}$ is well defined and, from part (a), also $\frac{1}{f(a+h)}$ is well defined for h small enough. Then

$$\lim_{h \to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h} = \lim_{h \to 0} \Big(\frac{f(a) - f(a+h)}{h} \cdot \frac{1}{f(a) f(a+h)} \Big) = -\frac{f'(a)}{f^2(a)},$$

where the first equality is obtained after elementary algebra and the second follows from the definition of the derivative at a and the fact that f is also continuous at a. Thus

$$\Big(\frac{1}{f}\Big)'(a) = -\frac{f'(a)}{f^2(a)} \ . \ \square$$

5, page 100. (a) Let $x \in \mathbf{R} \setminus 0$ arbitrary. The conditions to apply the Mean Value Theorem for f on the interval between 0 and x are satisfied, so there exists y (between 0 and x) such that

$$f(x) - f(0) = f'(y)(x - 0).$$

Thus, from the assumption, it follows that $f(x) - f(0) = 0$, for all $x \in \mathbf{R}$. $\square$

(b) The inequality trivially holds for $x = 0$ (it's actually equality in this case). Again let $x \in \mathbf{R} \setminus 0$ arbitrary and apply the Mean Value Theorem for f on the interval between 0 and x. There exists y such that $f(x) - f(0) = f'(y)(x - 0)$, and given the hypothesis in this case implies

$$|f(x) - 1| = |f'(y)||x| \leq |x|.$$

But by triangle inequality $|f(x)| - 1 \leq |f(x) - 1|$, so combining these we get $|f(x)| \leq |x| + 1$, $\forall x \in \mathbf{R}$. $\square$

(c) Let $a < b$ arbitrary. By the Mean Value Theorem for f on the interval $[a, b]$, there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. But by assumption $f'(c) \geq 0$, and $b - a > 0$, so it follows that $f(b) - f(a) \geq 0$. $\square$