Homework 3 - MAS 5145 Fall 2011 **NAME:** ______________________________

To receive credit you MUST SHOW ALL YOUR WORK.

1. (10 pts) Show that the vector space $M_{n,n}(\mathbf{R})$ of real $n \times n$ matrices can be decomposed as the direct sum $M_{n,n}(\mathbf{R}) = Sym_n \oplus ASym_n$, where Sym_n is the subspace of symmetric $n \times n$ matrices ($A^T = A$) and $ASym_n$ is the subspace of anti-symmetric $n \times n$ matrices ($A^T = -A$). What are the dimensions of these subspaces?
(For the last question, look at the first exercise in your previous homework and generalize.)

2. (15 pts) A linear operator $p : V \to V$ is called a *projector* of the vector space V if $p^2 = p$. We denote $p^2 = p \circ p$. Show that if p is a projector of V, then:

(a) $V = \mathrm{Im}p \oplus \mathrm{Ker}p$;

(b) the operator $q = Id_V - p$ is also a projector of V (Id_V denotes the identity of V) ;

(c) the operator $s = 2p - Id_V$ is an involutive automorphism of V; that is, you should show that $s^2 = Id_V$ and that s is an isomorphism from V to V.

3. (5 pts bonus) In this exercise $|A|$ denotes the cardinality of a set A. You can use the following known facts.

If $\mathcal{P}_0(A)$ denotes the set of **finite** subsets of A, then $|A| = |\mathcal{P}_0(A)|$ (i.e. there is a bijection between A and $\mathcal{P}_0(A)$).
If $\mathcal{P}(A)$ denotes the set of all subsets of A, then $|A| < |\mathcal{P}(A)|$ (i.e., there is an injection from A to $\mathcal{P}(A)$, but not the other way around).

Let V be an infinite dimensional vector space over the field $\mathbb{Z}_2 = \{0, 1\}$, with a basis $\mathcal{B}$. Denote by V^* the dual space of V. Prove that $|V| = |\mathcal{P}_0(\mathcal{B})| = |\mathcal{B}|$, whereas $|V^*| = |\mathcal{P}(\mathcal{B})|$. Thus $|V^*| > |V|$, so V^* cannot be isomorphic to V.