Homework 4 - MAS 5145 Fall 2011 **NAME:** ____________________

To receive credit you MUST SHOW ALL YOUR WORK.

1. (10 pts) Suppose V is an n-dimensional vector space over the field $\mathbb{K}$ and denote by $B(V \times V, \mathbb{K})$ the space of bilinear forms on V. Show that $B(V \times V, \mathbb{K})$ is a vector space isomorphic to the space of matrices $M_{n,n}(\mathbb{K})$. Moreover, through this isomorphism, symmetric bilinear forms correspond to symmetric matrices and anti-symmetric bilinear forms correspond to anti-symmetric matrices.

2. (20 pts) (a) The complex plane $\mathbb{C}$ can be thought as a 1-dimensional complex vector space, or as the 2-dimensional real vector space $\mathbb{R}^2$, with the identification

$$\mathbb{C} \ni (a + ib) \leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 .$$

Under this identification, determine the linear operator $J_0 : \mathbb{R}^2 \to \mathbb{R}^2$ that corresponds to the multiplication by i in $\mathbb{C}$. Show that $J_0^2 = -Id$ and show that the matrix of J_0 with respect to the standard basis in $\mathbb{R}^2$ is

$A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. With a slight abuse of notation we denote the matrix by J_0 as well.

Generalize to higher dimensions: under the identification

$$\mathbb{C}^n \ni \begin{pmatrix} a_1 + ib_1 \\ ... \\ a_n + ib_n \end{pmatrix} \leftrightarrow \begin{pmatrix} a_1 \\ b_1 \\ ... \\ a_n \\ b_n \end{pmatrix} \in \mathbb{R}^{2n} ,$$

determine the linear operator $J_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that corresponds to the multiplication by i in $\mathbb{C}^n$ and find its matrix with respect to the standard basis.

(b) Suppose V is a finite dimensional vector space and suppose J is an operator on V that satisfies $J^2 = -Id$. Such a J is called an (almost) complex structure on V. Show that $dim(V)$ must be even and show that there exists a basis of V, $\mathcal{B} = \{\mathbf{e}_1, \mathbf{f}_1, \mathbf{e}_2, \mathbf{f}_2, ..., \mathbf{e}_n, \mathbf{f}_n\}$, so that

$$[J]_{\mathcal{B}} = \begin{pmatrix} A_0 & 0 & ... & 0 \\ 0 & A_0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & A_0 \end{pmatrix} , \text{ where } A_0 \text{ is the } 2{\times}2 \text{ matrix from part (a) .}$$

3. (10 pts) Problem 14, page 67 textbook.