Name: Solution Key Panther ID: ____________

Exam 1 - MAC2311 - Spring 2015

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) These are True or False questions. No justification required. No partial credit. 2 points each.

(i) For all $a > 0, b > 0$, $\sqrt{a^2+b^2} = a+b$ True **False** $a^2+b^2 \neq (a+b)^2$

(ii) The equation $x^3 - 3x + 1 = 0$ has a solution in the interval $[1, 2]$. **True** False (I.V.T.)

(iii) If $\lim_{x\to 3} f(x) = 1$ and $\lim_{x\to 3} g(x) = -2$ then $\lim_{x\to 3}[f(x)-3g(x)] = 7$ **True** False (properties of limits)

(iv) If $x \neq 0$, $\frac{\sin x}{x} = 1$ True **False** $\lim_{x\to 0} \frac{\sin x}{x} = 1$, but without $\lim_{x\to 0}$ this is certainly false.

(v) The function $f(x) = \sec x$ is defined and is continuous for all real numbers x. True **False** $\sec x = \frac{1}{\cos x}$ is not defined at $\frac{(2k+1)\pi}{2}$.

(vi) The inverse function of $\sin x$ is $\frac{1}{\sin x}$. True **False** inverse function $\underline{\text{is not}}$ the reciprocal

2. (10 pts) Find all real solutions of the following equations (5 pts each):

(a) $2\cos x + 1 = 0 \iff \cos x = -\frac{1}{2}$ so $x = \frac{2\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$ (sol. in second quadrant)

or $x = \frac{4\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$ (sol. in third quadrant)

(b) $\log_{10}(x-15) - \log_{10} 2 = 3 \quad \underset{\text{property of log's}}{\iff}$

$\iff \log_{10}\left(\frac{x-15}{2}\right) = 3 \quad \underset{\text{def of } \log_{10}}{\iff} \quad \frac{x-15}{2} = 10^3$

Thus $x - 15 = 2 \times 1000$ so $\boxed{x = 2015}$!

3. (12 pts) An object is dropped from the top of a building. Its position $s(t)$ in feet above the ground t seconds after it was dropped is given by $s(t) = 80 - 16t^2$.

(a) (3 pts) When does the object hit the ground?

$t = ?$ for $s(t) = 0$ $\quad 0 = 80 - 16t^2 \Rightarrow 16t^2 = 80 \Rightarrow$

$\Rightarrow t^2 = 5 \Rightarrow \boxed{t = \sqrt{5}\ \text{s}}$

(b) (3 pts) Find the average velocity of the object during the first two seconds of its drop.

$$V_{\substack{ave \\ \text{when } t \in [0,2]}} = \frac{s(2) - s(0)}{2 - 0} = \frac{(80 - 16 \cdot 2^2) - (80 - 16 \cdot 0^2)}{2} = \frac{\cancel{80} - 64 - \cancel{80}}{2} = \boxed{-32\ \frac{ft}{s}}$$

(c) (6 pts) Use limits to find the instantaneous velocity of the object at 2 seconds.

$$V_{\substack{inst \\ \text{when } t=2}} = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h \to 0} \frac{(80 - 16(2+h)^2) - (80 - 16 \cdot 2^2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{80} - 16(4 + 4h + h^2) - \cancel{80} + 64}{h} = \lim_{h \to 0} \frac{\cancel{-64} - 64h - 16h^2 \cancel{+64}}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}(64 + 16h)}{\cancel{h}\,1} = \boxed{-64\ \frac{ft}{s}}$$

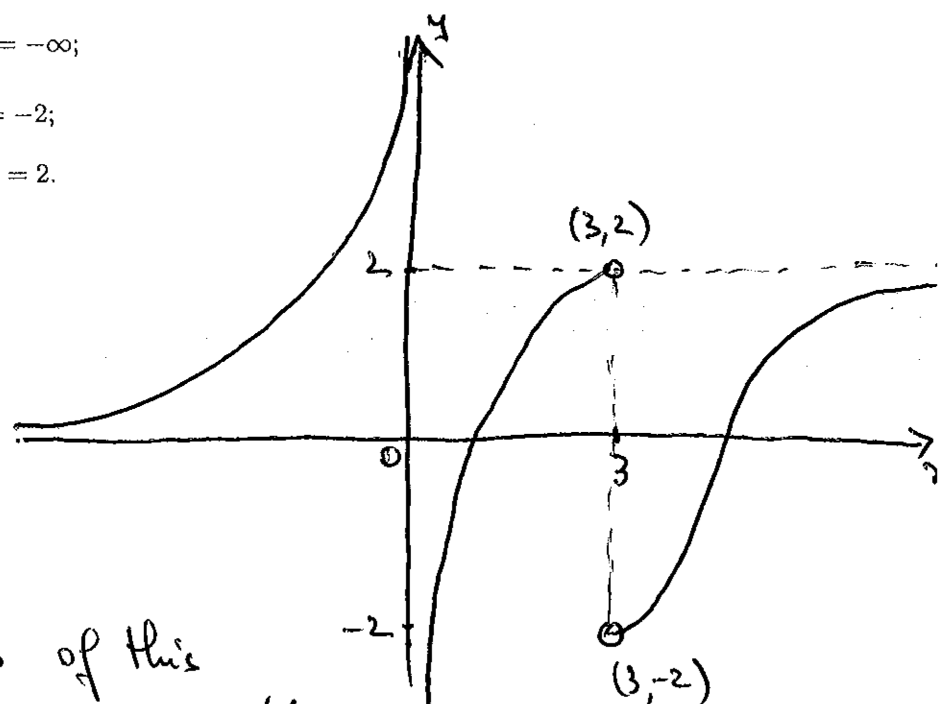
4. (12 pts) Sketch a graph of ONE function $f(x)$ satisfying all of the following conditions.

(i) The function is defined and is continuous for all real x, except $x = 0$ and $x = 3$; the function is not defined at $x = 0$ and $x = 3$.

(ii) $\lim_{x \to 0^-} f(x) = +\infty$ and $\lim_{x \to 0^+} f(x) = -\infty$;

(iii) $\lim_{x \to 3^-} f(x) = 2$ and $\lim_{x \to 3^+} f(x) = -2$;

(iv) $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to +\infty} f(x) = 2$.



variations of this are, of course, possible as long as the main features are satisfied

5. (30 pts) Find the following limits (5 pts each). If the limit is infinite or does not exist, specify so.

(a) $\lim_{x\to 2} \frac{x^2-5x+6}{4x-x^3} \overset{\frac{0}{0}}{=} \lim_{x\to 2} \frac{(x-2)(x-3)}{x(4-x^2)} =$

$= \lim_{x\to 2} \frac{(x-2)(x-3)}{x(2-x)(2+x)} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x-3)}{-x\cdot\cancel{(x-2)}\cdot(2+x)} =$

$= \frac{-1}{(-2)\cdot 4} = \boxed{\frac{1}{8}}$

(b) $\lim_{x\to +\infty} \frac{x^2-5x+6}{4x-x^3} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to+\infty} \frac{\cancel{x^2}\left(1-\frac{5}{x}+\frac{6}{x^2}\right)}{x^{\cancel{3}}\left(\frac{4}{x^2}-1\right)}$

$= \lim_{x\to+\infty} \frac{\left(1-\frac{5}{x}+\frac{6}{x^2}\right)}{x\left(\frac{4}{x}-1\right)} = \frac{1}{-\infty} = \boxed{0}$

(c) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \overset{\frac{0}{0}}{=}$ (multiply with conjugate)

$= \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} =$

$= \lim_{x\to 4} \frac{\cancel{(x-4)}\cdot 1}{\cancel{(x-4)}(\sqrt{x}+2)} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$

(d) $\lim_{x\to 1} \frac{x^2-2x+1}{|x-1|} \overset{\frac{0}{0}}{=} \lim_{x\to 1} \frac{(x-1)^2}{|x-1|}$

you may compute the two one-sid limits, but you can avoid this if you notice that

$\frac{(x-1)^2}{|x-1|} = \frac{|x-1|^2}{|x-1|} = |x-1|$

so $\lim_{x\to 1} \frac{(x-1)^2}{|x-1|} = \lim_{x\to 1} |x-1| = \boxed{0}$.

(d) $\lim_{x\to 0} \frac{\sin(6x)}{x+\sin(2x)} \overset{\frac{0}{0}}{=}$

$= \lim_{x\to 0} \frac{\frac{\sin(6x)}{6x}\cdot 6x}{x+\frac{\sin(2x)}{2x}\cdot 2x} =$

$= \lim_{x\to 0} \frac{\frac{\sin(6x)}{6x}\cdot 6\cancel{x}}{\cancel{x}\left[1+2\cdot\frac{\sin(2x)}{2x}\right]} = \frac{1\cdot 6}{1+2} =$

$= \frac{6}{3} = \boxed{2}$

(e) $\lim_{x\to 0^-} \cot x = \lim_{x\to 0^-} \frac{\cos x}{\sin x} = \frac{1}{0^-} = \boxed{-\infty}$

6. (10 pts) Find, if possible, a value for the constant k which will make the function $g(x)$ continuous everywhere. If you think there is no such k, justify why not.

$$g(x) = \begin{cases} \frac{\tan(kx)}{x} & \text{if } x < 0 \\ kx + 3 & \text{if } x \geq 0 \end{cases}$$

The problem point is $x=0$.
For $g(x)$ to be continuous at $x=0$,
we want $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = g(0)$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{\tan(kx)}{x} = \lim_{x\to 0^-} \frac{k\cdot\tan(kx)}{kx} = \boxed{k}$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (kx+3) = k\cdot 0 + 3 = \boxed{3 = g(0)}$$

Thus, if $\boxed{k=3}$, $g(x)$ is continuous at $x=0$ and everywhere.

7. (a) (3 pts) Write the general (ϵ, δ) definition for $\lim_{x\to a} f(x) = L$.

For any $\varepsilon > 0$ we can find $\delta > 0$ so that if $|x-a| < \delta$ and $x \neq a$ then $|f(x) - L| < \varepsilon$.

Choose ONE of the parts (b) and (c). Only ONE will receive credit. Note the different point values.

(b) (7 pts) Use the (ϵ, δ) definition to prove $\lim_{x\to 2}(2x-7) = -3$.

(c) (12 pts) Use the (ϵ, δ) definition to prove $\lim_{x\to 2}(2x^2-7) = 1$.

(b) $|f(x)-L| = |2x-7-(-3)| = |2x-4| = 2\cdot|x-2|$

Given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{2}$.

Then if $|x-2| < \delta \Rightarrow |f(x)-L| = 2\cdot|x-2| < 2\cdot\delta = 2\cdot\frac{\varepsilon}{2}$

(c) $|f(x)-L| = |2x^2-7-1| = 2|x^2-4| = 2\cdot|x-2|\cdot|x+2|$

Preliminary choice: $\delta \leq 1$

Then if $|x-2| < \delta \leq 1 \Rightarrow |x-2| < 1 \Rightarrow -1 < x-2 < 1 \Rightarrow$ Add 4 to all sides $3 < x+2 < 5$

Thus, $\boxed{\text{if } |x-2| < \delta \leq 1 \Rightarrow |x+2| < 5}$, so

$|f(x)-L| = 2\cdot|x-2|\cdot|x+2| < 2\cdot|x-2|\cdot 5 < 10\delta$ if $|x-2| < \delta \leq 1$

Given $\varepsilon > 0$

Final choice for $\delta = \min\left(1, \frac{\varepsilon}{10}\right)$. Note that with this choice $\delta \leq 1$ and $\delta \leq \frac{\varepsilon}{10}$

Then if $|x-2| < \delta \Rightarrow$

$\Rightarrow |f(x)-L| = 2\cdot|x-2|\cdot|x+2| < 10\delta \leq 10\cdot\frac{\varepsilon}{10}$ as on the line above

8. (10 pts) Choose ONE of the following:

(a) State and prove the quadratic formula.

(b) Assuming the inequality $\sin x \le x \le \tan x$ for $x \in [0, \pi/2)$, prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

See notes on text.