NAME: Solution Key Panther ID: ____________

Exam 2 - MAC 2311 Spring 2015

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (30 pts) Find dy/dx. Simplify when possible (6 pts each):

(a) $y = \frac{x^3}{3} - 2\sqrt{x} + 3\pi^{19} = \frac{1}{3}x^3 - 2x^{\frac{1}{2}} + \underbrace{3\pi^{19}}_{\text{constant}}$

$y' = \frac{1}{3}\cdot 3x^2 - 2\cdot\frac{1}{2}x^{-\frac{1}{2}} + 0$

$\boxed{y' = x^2 - \frac{1}{\sqrt{x}}}$

(b) $y = x^2 e^{-3x}$

Product Rule $y' = (x^2)'e^{-3x} + x^2(e^{-3x})'$

$y' = 2xe^{-3x} + x^2e^{-3x}\cdot(-3)$

$\boxed{y' = e^{-3x}\cdot x\cdot(2-3x)}$

(c) $y = \sqrt{1+\sec^2 x} = (1+\sec^2 x)^{\frac{1}{2}}$

Chain Rule

$y' = \frac{1}{2}(1+\sec^2 x)^{-\frac{1}{2}}\cdot(1+\sec^2 x)'$

Chain Rule

$y' = \frac{1}{\cancel{2}}(1+\sec^2 x)^{-\frac{1}{2}}\cdot\cancel{2}\sec x\cdot(\sec x)'$

$y' = \frac{1}{\sqrt{1+\sec^2 x}}\cdot\sec x\cdot(\sec x)\cdot\tan x$

$\boxed{y' = \frac{\sec^2 x\cdot\tan x}{\sqrt{1+\sec^2 x}}}$

(d) $y = \arctan(\ln x)$

Chain Rule

$y' = \frac{1}{1+(\ln x)^2}\cdot(\ln x)'$

$\boxed{y' = \frac{1}{1+(\ln x)^2}\cdot\frac{1}{x} = \frac{1}{x(1+(\ln x)^2)}}$

(e) $y = (x^2+1)^x$

Logarithmic Differentiation

$\ln y = \ln\left((x^2+1)^x\right) = x\cdot\ln(x^2+1)$ | Take $\frac{d}{dx}$

$(\ln y)' = (x\cdot\ln(x^2+1))'$ Product Rule

Chain Rule $\frac{1}{y}\cdot y' = 1\cdot\ln(x^2+1) + x\cdot\frac{1}{x^2+1}\cdot 2x$

so $y' = y\left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)$ so $\boxed{y' = (x^2+1)^x\left[\ln(x^2+1) + \frac{2x^2}{x^2+1}\right]}$

2. (8 pts) If $f(x) = \cos(3x)$, determine $f^{(2015)}(x)$.

$f^{(0)}(x) = f(x) = \cos(3x)$,

$f^{(1)}(x) = f'(x) = (\cos(3x))' = -3\sin(3x)$

$f^{(2)}(x) = f''(x) = -3\cdot 3\cos(3x) = -3^2\cos(3x)$

$f^{(3)}(x) = 3^3\sin(3x)$

$f^{(4)}(x) = 3^4\cos(3x)$

so the powers of 3 increase every time and the pattern $\cos, -\sin, -\cos, \sin$ repeats every four

Since $2015 \div 4$ has remainder 3,

$$\boxed{f^{(2015)}(x) = 3^{2015}\sin(3x)}$$

3. (10 pts) The function $h(x)$ is given by $h(x) = \frac{1+x^2}{f(x)}$. Given that $f(2) = 1$ and $f'(2) = 5$, find

(a) (3 pts) $h(2)$

$$h(2) = \frac{1+2^2}{f(2)} = \frac{5}{1} = \boxed{5}$$

(b) (7 pts) $h'(2)$

Quotient Rule

$$h'(x) = \left(\frac{1+x^2}{f(x)}\right)' = \frac{2x\cdot f(x) - (1+x^2)f'(x)}{(f(x))^2}$$

$$h'(2) = \frac{2\cdot 2\cdot f(2) - (1+2^2)\cdot f'(2)}{(f(2))^2} = \frac{4\cdot 1 - 5\cdot 5}{1^2} = \boxed{-21}$$

4. (12 pts) Find the equation of the tangent line to the curve $3x^2 + 2xy^3 = 5y^2$ at the point $(1, 1)$.

We find $\frac{dy}{dx}$ using implicit differentiation:

$(3x^2 + 2x\cdot y^3)' = (5y^2)'$, where "'" denotes derivative with respect to x and we think of y as a function of x.

$$6x + 2\cdot y^3 + 2x\cdot 3y^2\cdot y' = 10y\cdot y'$$

$$6x + 2y^3 = y'(10y - 6xy^2) \Rightarrow y' = \frac{dy}{dx} = \frac{6x+2y^3}{10y-6xy^2}$$

Thus the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{6\cdot 1 + 2\cdot 1^3}{10\cdot 1 - 6\cdot 1} = \frac{8}{4} = 2$$

From the point-slope formula, the tangent line is

$\boxed{y - 1 = 2\cdot(x-1)}$ or $\boxed{y = 2x - 1}$

5. (12 pts) Show that $y = x\cos(3x)$ is a solution to $y'' + 9y = -6\sin(3x)$.

We should compute y''.

$$y' = (x\cdot\cos(3x))' = 1\cdot\cos(3x) + x\cdot(-\sin(3x))\cdot 3$$

so $y' = \cos(3x) - 3x\sin(3x)$

$$y'' = (y')' = (\cos(3x))' - (3x\sin(3x))'$$

$$y'' = -\sin(3x)\cdot 3 - (3\sin(3x) + 3x\cos(3x)\cdot 3)$$

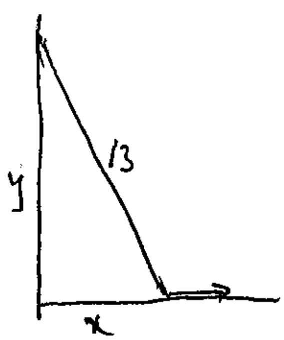
$$y'' = -3\sin(3x) - 3\sin(3x) - 9x\cos(3x) = -6\sin(3x) - 9x\cos(3x)$$

Now compute

$$y'' + 9y = -6\sin(3x) - \cancel{9x\cos(3x)} + \cancel{9x\cos(3x)} = -6\sin(3x)$$

so, indeed, $y = x\cos(3x)$ is a solution for $y'' + 9y = -6\sin(3x)$

6. (12 pts) A 13ft ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 0.12 ft/s, how fast will the top of the ladder be moving down the wall when it is 12ft above the ground?



$x(t)$, $y(t)$

given $\frac{dx}{dt} = 0.12\,\frac{ft}{s}$ find $\frac{dy}{dt} = ?$

Pythagora: $x^2 + y^2 = 13^2$ | Take $\frac{d}{dt}$

$$2x\cdot\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow 2y\frac{dy}{dt} = -2x\frac{dx}{dt} \Rightarrow$$

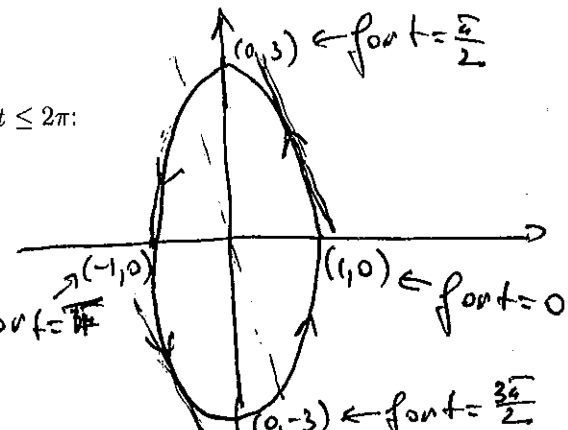
$$\Rightarrow \frac{dy}{dt} = -\frac{x\frac{dx}{dt}}{y}$$

when $y = 12$, $x^2 + 12^2 = 13^2 \Rightarrow x^2 = 169 - 144 = 25 \Rightarrow x = 5$

Thus $\frac{dy}{dt} = -\frac{5\times 0.12}{12} = \frac{-5\times\cancel{12}}{100\times\cancel{12}} = \boxed{-0.05\,\frac{ft}{s}}$

7. (14 pts) Given the parametric curve $x = \cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$:

(a) (6 pts) Sketch the curve in the xy plane, marking the coordinates of axis intercepts and indicating orientation.



Eliminate the parameter, using

$\cos^2 t + \sin^2 t = 1$

$x^2 + \left(\frac{y}{3}\right)^2 = 1 \leftarrow$ ellipse

(b) (8 pts) Find the coordinates of a point on the curve (if any) with the property that the tangent line to the curve at that point is parallel to the line $y = -\sqrt{3}x$.

We want to find a point on the curve where the slope of the tangent line is equal to $-\sqrt{3}$ (parallel lines have equal slopes)

$$m_{tan} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t}{-\sin t} \leftarrow \text{slope of tangent line at an arbitrary point}$$

We solve $\frac{3\cos t}{-\sin t} = -\sqrt{3} \Rightarrow -3\cot t = -\sqrt{3} \Rightarrow$

$\Rightarrow \cot t = \frac{\sqrt{3}}{3} \Rightarrow \frac{1}{\tan t} = \frac{\sqrt{3}}{3} \Rightarrow \tan t = \frac{3}{\sqrt{3}} = \sqrt{3} \Rightarrow t = \frac{\pi}{3}$ and $\frac{4\pi}{3}$

So there are two points $P_1\left(\cos\frac{\pi}{3}, 3\sin\frac{\pi}{3}\right)$ or $P_1\left(\frac{1}{2}, \frac{3\sqrt{3}}{2}\right)$ and $P_2\left(\cos\frac{4\pi}{3}, 3\sin\frac{4\pi}{3}\right)$ or $P_2\left(-\frac{1}{2}, -\frac{3\sqrt{3}}{2}\right)$

8. (12 pts) Choose ONE:

(a) State and prove the formula for the derivative of a product of two functions.

(b) Find, with proof, the formula for $(\arcsin x)'$.

see notes or textbook