Exam 3- MAC 2311, Sp'15 **NAME:** Solution Key **Panther ID:** ____________

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (6 pts) Solve the initial value problem:

$$\frac{dy}{dx} = 3\sec^2 x + 1, \quad y(0) = 2$$

$y(x) = \int (3\sec^2 x + 1)\,dx = 3\tan x + x + C$

$y(0) = 2 \Rightarrow 2 = 3 \cdot 0 + 0 + C \Rightarrow C = 2$

So $\boxed{y(x) = 3\tan x + x + 2}$

2. (24 pts) Find each indicated antiderivative:

(a) $\int (2x^2 + 2^x - \frac{1}{x^2})\,dx = \int (2x^2 + 2^x - x^{-2})\,dx =$

$= \frac{2x^3}{3} + \frac{1}{\ln 2} \cdot 2^x + x^{-1} + C$

$= \frac{2}{3}x^3 + \frac{1}{\ln 2} \cdot 2^x + \frac{1}{x} + C$

(b) $\int x^2\sqrt{x^3+9}\,dx = \int \sqrt{w} \cdot \frac{1}{3}\,dw = *$

sub. $w = x^3 + 9$

$dw = 3x^2\,dx$

$\frac{1}{3}dw = x^2\,dx$

$* = \frac{1}{3}\int w^{\frac{1}{2}}\,dw = \frac{1}{3} \cdot \frac{2}{3} w^{\frac{3}{2}} + C$

$\boxed{= \frac{2}{9}(x^3+9)^{\frac{3}{2}} + C}$

(c) $\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{du}{1+u^2} =$

sub. $u = e^x$

$du = e^x\,dx$

$= \arctan(u) + C =$

$\boxed{= \arctan(e^x) + C}$

(d) $\int \frac{1}{x \ln x}\,dx =$

$w = \ln x$

$dw = \frac{1}{x}dx$

$= \int \frac{1}{w}\,dw = \ln|w| + C = \boxed{\ln|\ln x| + C}$

3. (10 pts) Use an appropriate local linear approximation to estimate $\sqrt[3]{8.12}$ without a calculator. Be sure to specify the function and the point you are using for the local linear approximation.

Loc. lin. approx. for $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ at $x_0 = 8$

$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$

$f(8) = \sqrt[3]{8} = 2$

$f'(x) = (x^{\frac{1}{3}})' = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3(\sqrt[3]{x})^2}$

$f'(8) = \frac{1}{3(\sqrt[3]{8})^2} = \frac{1}{12}$

$\Rightarrow \sqrt[3]{x} \approx 2 + \frac{1}{12}(x-8)$ for x near 8

So $\sqrt[3]{8.12} \approx 2 + \frac{1}{12}(8.12 - 8)$

$\boxed{\sqrt[3]{8.12} \approx 2.01}$

4. (10 pts) Find the absolute minimum m and the absolute maximum M for the function $f(x) = x^3 - x^4$ over the interval $[-1, \frac{1}{2}]$ (if they exist) and state the values of x where the absolute extrema occur.

$f'(x) = 3x^2 - 4x^3 = x^2(3 - 4x)$

Critical points $x = 0$, $x = \frac{3}{4}$. But $x = \frac{3}{4}$ is not in the interval $[-1, \frac{1}{2}]$ so we don't consider it.

Candidates for abs. min/max.

$x_1 = -1$, $x_2 = 0$, $x_3 = \frac{1}{2}$

$f(-1) = (-1)^3 - (-1)^4 = -2$ $\qquad f(0) = 0$ $\qquad f(\frac{1}{2}) = (\frac{1}{2})^3 - (\frac{1}{2})^4 = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$

So abs. min is reached at $x = -1$ and $m = f(-1) = -2$

and

abs. max is reached at $x = \frac{1}{2}$ and $M = f(\frac{1}{2}) = \frac{1}{16}$

5. (10 pts) Derive the equations for the velocity $v(t)$ and position $s(t)$ at time time t for an object that is moving on a straight line with constant acceleration a. Assume the initial position is s_0 and the initial velocity is v_0.

$a(t) = a$

$v(t) = \int a \cdot dt = at + c$

$v_0 = v(0) = 0 + c$

$\Rightarrow \boxed{v(t) = at + v_0}$

$s(t) = \int v(t)dt = \int (at + v_0)dt = \frac{at^2}{2} + v_0 t + \tilde{c}$

$s_0 = s(0) = \tilde{c}$

So $\boxed{s(t) = \frac{at^2}{2} + v_0 t + s_0}$

6. (20 pts) Give a complete graph of the function $f(x) = \frac{x^2-1}{(x+2)^2}$. Your work should include: the domain of the function, equations of eventual asymptotes (vertical or/and horizontal) justified with limits, a sign chart for the derivative and the second derivative, the location and nature of the critical points (if any), location of inflection points (if any), coordinates for the axis intercepts. To save you time, here is the second derivative $f''(x) = \frac{-8x+2}{(x+2)^4}$. The first derivative you have to compute on your own. (Please, do it well!).

(pts) Domain: all reals, except $x = -2$

2pts) $\lim_{x\to -2^+} \frac{x^2-1}{(x+2)^2} = \lim_{x\to -2^-} \frac{x^2-1}{(x+2)^2} = \frac{(-2)^2-1}{0^+} = \frac{3}{0^+} = +\infty$

So $x=-2$ is a vertical asymptote and $\lim_{x\to -2} f(x) = +\infty$

(2pts) Horiz. asymptote / end behaviour

$\lim_{x\to +\infty} \frac{x^2-1}{(x+2)^2} = 1$ (l'H or junk rule) so $y=1$ is a horiz. asymptote both when $x\to -\infty$ and $x\to +\infty$

2pts) $f'(x) = \left(\frac{x^2-1}{(x+2)^2}\right)' = \frac{2x\cdot(x+2)^2 - (x^2-1)\cdot 2(x+2)}{(x+2)^4} = \frac{2(x+2)[\cancel{x^2}+2x-\cancel{x^2}+1]}{(x+2)^4}$

2pts) Thus $f'(x) = \frac{2(2x+1)}{(x+2)^3}$ $\quad f'(x)=0 \iff x=-\frac{1}{2}$ ← critical pt.

pt) $f''(x) = \frac{-8x+2}{(x+2)^4}$ so $f''(x)=0 \iff x=\frac{1}{4}$ ← possible inflection pt.

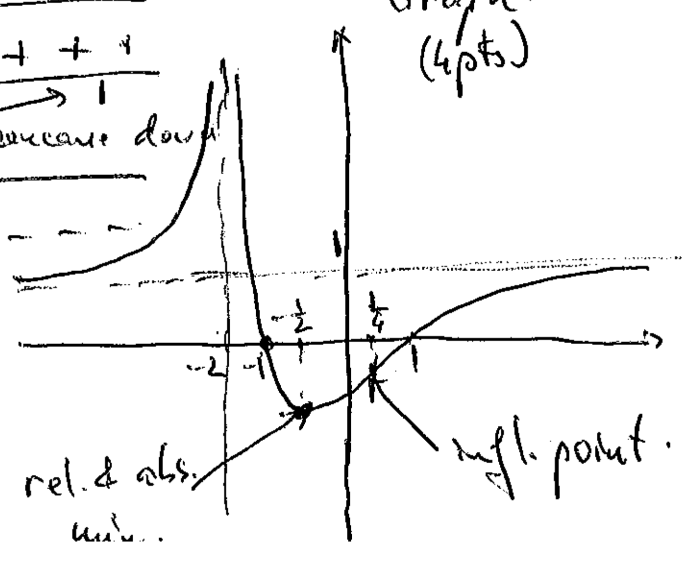
Test value

rel. min. $f(-\frac{1}{2}) = \frac{\frac{1}{4}-1}{(-\frac{1}{2}+2)^2} < 0$

pts) Sign chart

x	$-\infty$		-2		$-\frac{1}{2}$	0	$\frac{1}{4}$		$+\infty$
$f'(x)$		+ + + + + +	‖	− − −	0	+ + +	+	+ + + +	
$f(x)$	1	concave up ↗	$+\infty$ ‖ $+\infty$	concave up ↘		↗		concave down ↗	1
$f''(x)$		+ + + + + + +	‖	+ + + + +			0	− − − − −	

Graph. (4pts)



pts) y-intercept $f(0) = \frac{-1}{(2)^2} = -\frac{1}{4}$

x-intercepts $f(x)=0 \iff x^2=1 \iff x=\pm 1$

7. (16 pts) Compute each of the following limits:

(a) $\lim_{x\to 0} \dfrac{e^{3x}-1}{\sin(2x)} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to 0} \dfrac{3\cdot e^{3x}}{\cos(2x)\cdot 2} = \boxed{\dfrac{3}{2}}$

(b) $\lim_{x\to +\infty} [\cos(3/x)]^{x^2} = \lim_{x\to+\infty} e^{\ln\left([\cos(\frac{3}{x})]^{x^2}\right)}$

$(\cos(0))^{\infty} = 1^{\infty}$

$= e^{\lim_{x\to+\infty} x^2 \cdot \ln[\cos(\frac{3}{x})]}$ $\quad \infty\cdot 0$

$\lim_{x\to+\infty} \dfrac{\ln[\cos(\frac{3}{x})]}{\frac{1}{x^2}} \overset{\frac{0}{0}}{\underset{l'H}{=}}$

$= \lim_{x\to+\infty} \dfrac{\frac{1}{\cos(\frac{3}{x})}\cdot\left(-\sin(\frac{3}{x})\right)\cdot\left(-\frac{3}{\cancel{x^2}}\right)}{-\frac{2}{\cancel{x^3}}\,x} =$

$= -3\lim_{x\to+\infty} \dfrac{\sin(\frac{3}{x})}{\frac{2}{x}} \cdot \dfrac{1}{\cos(\frac{3}{x})} = -3\cdot\dfrac{3}{2} = -\dfrac{9}{2}$

So $\lim_{x\to+\infty} [\cos(\frac{3}{x})]^{x^2} = \boxed{e^{-\frac{9}{2}}}$

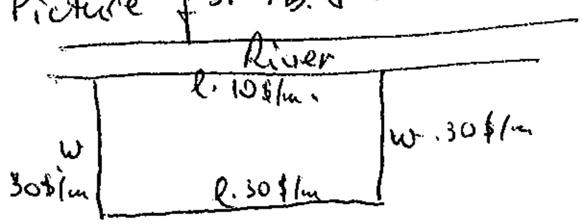
easier is to do here the sub. $w = \frac{1}{x}$. Then $x\to+\infty \Rightarrow w\to 0^+$

$\lim_{w\to 0^+} \dfrac{\ln[\cos(3w)]}{w^2} \overset{\frac{0}{0}}{\underset{l'H}{=}}$

$= \lim_{w\to 0^+} \dfrac{\frac{1}{\cos(3w)}\cdot(-\sin(3w))\cdot 3}{2w} =$

$= -\dfrac{3}{2}\lim_{w\to 0^+} \underbrace{\dfrac{\sin(3w)}{w}}_{\to 3} \cdot \underbrace{\dfrac{1}{\cos(3w)}}_{\to 1} = -\dfrac{3}{2}\cdot 3 = \boxed{-\dfrac{9}{2}}$

Picture for Pb. 8.



8. (14 pts) Suppose you are allowed to choose a rectangular plot of land along a (straight) river. The rectangular plot is to have an area of 3000 square meters. You are required to fence in your land using two kinds of fencing. Three of the four sides will use heavy-duty fencing selling for \$30 per meter while the remaining side (along the river) will use standard fencing selling for \$10 per meter. How should you choose the dimensions of your plot of land in order to minimize the cost of fencing? (It's OK if your result contains square-roots.)

Area $\quad A = l\cdot w = 3000 \Rightarrow l = \dfrac{3000}{w}$

Cost $\quad C = l\cdot 10 + l\cdot 30 + 2w\cdot 30 = 40l + 60w$

$C(w) = 40\times\dfrac{3000}{w} + 60w = 60w + \dfrac{120{,}000}{w} \qquad w\in(0,+\infty)$

$C'(w) = 60 - \dfrac{120{,}000}{w^2} \Rightarrow C'(w) = 0 \to 60 = \dfrac{120{,}000}{w^2} \Rightarrow w^2 = 2000 \Rightarrow w = \sqrt{2000}$

$w = \sqrt{2000}$ is an abs. min. since $C''(w) = \dfrac{360{,}000}{w^3} > 0$

[illegible] 3000