Name: Solution Key Panther ID: ____________

Exam 1 Calculus II Fall 2019

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. Electronic devices (cell phones, calculators of any kind, etc.) should NOT be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
Violations of any type of this rule will lead to a score of zero on this exam, possibly an automatic grade F for the course and a report for academic misconduct.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) Quick answers questions (3 pts each). Answer and briefly indicate what you are using in each case.

(a) Suppose that oil is leaking into the ocean from a damaged tanker at a rate of $r(t)$ gallons per hour, where t is the time in hours since the accident occurred. In one sentence, explain what the integral $\int_{24}^{48} r(t)\, dt$ represents in practical terms.

$\int_{24}^{48} r(t)dt$ gives the total amount of oil that leaks in the ocean in the time interval $24 \le t \le 48$ hours since the accident

(b) If $\int_1^2 f(x)\, dx = -3$, $\int_1^5 f(x)\, dx = 4$, then $\int_2^5 f(x)\, dx = \int_1^5 f(x)dx - \int_1^2 f(x)dx = 4-(-3) = \boxed{7}$

(c) $\frac{d}{dx}\Big(\int_1^x \sin(t^2)\, dt\Big) = \sin(x^2)$ (by FTC)

(d) $\frac{d}{dx}\Big(\int_{3x}^{x^2} \sin(t^2)\, dt\Big) = \sin\left((x^2)^2\right)\cdot 2x - \sin\left((3x)^2\right)\cdot 3 =$

$= \sin(x^4)\cdot 2x - 3\sin(9x^2)$ (by Leibnitz's Rule)

2. (8 pts) Find the values of each of the following sums (OK to leave your answer as a product):

(a) (4 pts) $\sum_{k=1}^{1000}\left((k+1)^2-k^2\right)$

Solution 1: Observe that the sum is telescopic

$$\sum_{k=1}^{1000}\left((k+1)^2-k^2\right) = (\cancel{2^2}-1^2)+(\cancel{3^2}-\cancel{2^2})+(\cancel{4^2}-\cancel{3^2})+\dots+(1001^2-\cancel{1000^2})$$

$$= 1001^2-1^2 = (1001-1)(1001+1)$$

$$= 1000\cdot 1002 = \boxed{1002000}$$

Solution 2

$$\sum_{k=1}^{1000}\left((k+1)^2-k^2\right) = \sum_{k=1}^{1000}\left(\cancel{k^2}+2k+1-\cancel{k^2}\right) =$$

$$= 2\sum_{k=1}^{1000}k + \sum_{k=1}^{1000}1 \underset{\text{Gauss's formula}}{=} \cancel{2}\cdot\frac{1000\cdot 1001}{\cancel{2}} + 1000 = 1000\cdot(1001+1) = \boxed{1002000}$$

(b) (4 pts) 9+19+29+39+ ... +989+999 = $(10-1)+(20-1)+(30-1)+\dots+(990-1)+(1000-1)$

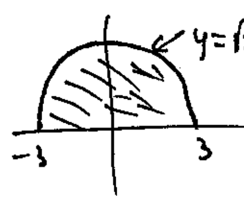
$$= \sum_{k=1}^{100}(10k-1) = 10\sum_{k=1}^{100}k - \sum_{k=1}^{100}1 \underset{\text{Gauss's formula}}{=} 10\cdot\frac{100\cdot 101}{2} - 100$$

$$= 100\left(\frac{10}{2}\cdot 101 - 1\right) = 100\cdot 504 = \boxed{50400}$$

3. (8 pts) Find the average value of the function $f(x)=\sqrt{9-x^2}$ over the interval $[-3,3]$.
(Hint: What is the graph of $y=\sqrt{9-x^2}$?)

$$av(f) = \frac{\int_{-3}^{3}\sqrt{9-x^2}\,dx}{3-(-3)}$$

but $\int_{-3}^{3}\sqrt{9-x^2} = \text{shaded area} = \frac{\pi\cdot 3^2}{2} = \frac{9\pi}{2}$



$y=\sqrt{9-x^2}$ ← graph is a semi-circle

$$av(f) = \frac{\frac{9\pi}{2}}{6} = \frac{9\pi}{12} = \boxed{\frac{3\pi}{4}}$$

4. (28 pts) Compute each integral (7 pts each):

(a) $\int_1^4 \left(\frac{1}{3}+2\sqrt{x}\right)\,dx$ $= \left(\frac{1}{3}x + 2\cdot\frac{2}{3}x^{\frac{3}{2}}\right)\Big|_{x=1}^{x=4} =$

$$= \frac{1}{3}x\Big|_{x=1}^{x=4} + \frac{4}{3}x^{\frac{3}{2}}\Big|_{x=1}^{x=4} =$$

$$= \frac{1}{3}(4-1) + \frac{4}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) =$$

$$4^{\frac{3}{2}} = \left(\sqrt{4}\right)^3 = 2^3 = 8$$

$$= \frac{1}{3}\cdot 3 + \frac{4}{3}(8-1) = 1 + \frac{28}{3} = \boxed{\frac{31}{3}}$$

(b) $\int_0^1 \frac{6x}{(5+x^2)^2}\,dx$ $= \int_{w=5}^{w=6} \frac{3\,dw}{w^2} = 3\int_{w=5}^{w=6} w^{-2}\,dw$

$w = 5 + x^2$

$dw = 2x\,dx$

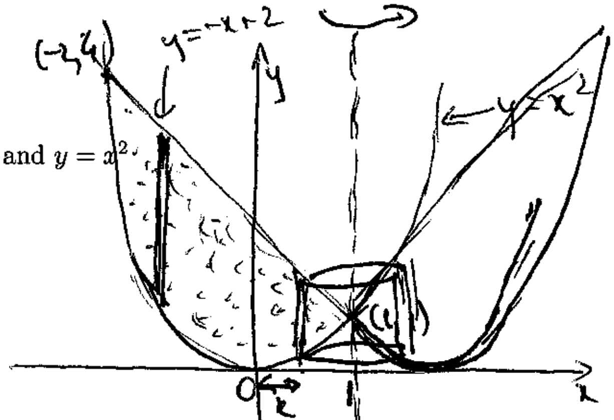
$$= -3w^{-1}\Big|_{w=5}^{w=6} = -3\left(\frac{1}{6}-\frac{1}{5}\right) = -3\left(-\frac{1}{30}\right) = \boxed{\frac{1}{10}}$$

(c) $\int_{1}^{e^2} \frac{(\ln x)^3}{2x}\,dx \underset{\substack{w=\ln x \\ dw=\frac{1}{x}dx}}{=} \int_{w=0}^{w=2} \frac{w^3}{2}\,dw = \frac{1}{2}\cdot\frac{w^4}{4}\Big|_{w=0}^{w=2}$

$$= \frac{2^4}{8} = \boxed{2}$$

(d) $\int_0^{\pi/4} e^{\tan x}\sec^2 x\,dx \underset{\substack{w=\tan x \\ dw=\sec^2 x\,dx}}{=} \int_{w=0}^{w=1} e^w\,dw = \left(e^w\right)\Big|_{w=0}^{w=1} = e^1 - e^0 =$

$$= \boxed{e-1}$$



5. Consider the region bounded between $y = -x + 2$ and $y = x^2$.

(a) (8 pts) Find the area of the region.

Full computation is required for this part.

Intersection pts:

$$\begin{cases} y = -x+2 \\ y = x^2 \end{cases} \Rightarrow x^2 = -x+2 \Leftrightarrow x^2+x-2=0 \Leftrightarrow (x+2)(x-1)=0 \Leftrightarrow x=-2 \text{ or } x=1$$

$$A = \int_?^? l_{strip} \cdot Th_{strip} = \int_{x=-2}^{x=1} (-x+2-x^2)\,dx = \left(-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right)\Big|_{x=-2}^{x=1} =$$

$Th_{strip} = dx$

$l_{strip} = y_{line} - y_{curve} = -x+2-x^2$

$$= -\frac{1}{3} - \frac{1}{2} + 2 - \left(\frac{8}{3} - \frac{4}{2} - 4\right) = -\frac{9}{3} + \frac{3}{2} + 6 = 3 + \frac{3}{2} = \boxed{\frac{9}{2}}$$

(b) (8 pts + 6 bonus) The region is now rotated around the vertical line $x = 1$. Set up an integral (or integrals) to represent the volume of the solid obtained. Computation of the integral is NOT required, just the set up is. Method of cylindrical shells is recommended, but a correct solution with the cross-section method also receives full credit. Actually, you'll receive up to 6 bonus points for solutions with BOTH methods.

Sol. with cyl. shells

$$V = \int_?^? 2\pi R_{shell} \cdot h_{shell} \cdot Th_{shell} = \boxed{2\pi \int_{x=-2}^{x=1} (1-x)(-x+2-x^2)\,dx}$$

$Th_{shell} = dx$

$h_{shell} = y_{line} - y_{curve} = -x+2-x^2$

$R_{shell} = 1-x$

Solution with cross-section method

$$V = \int_?^? A_{slice} \cdot Th_{slice}$$

$Th_{slice} = dy$

$A_{slice} = \pi(R^2 - r^2)$

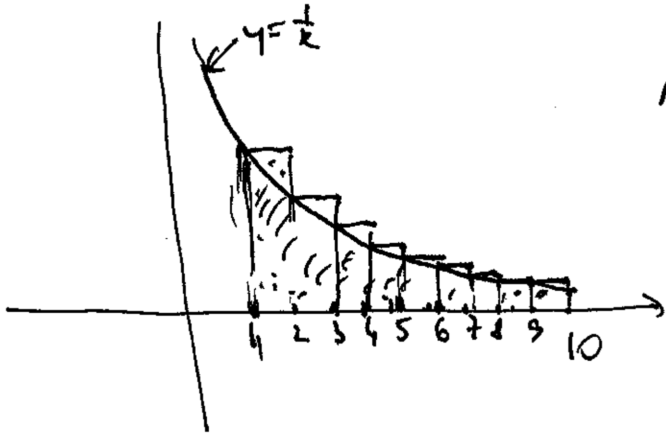
$R = 1-(-\sqrt{y}) = 1+\sqrt{y}$

$r = 1-\sqrt{y}$ ← if $0<y<1$

$r = 1-(2-y) = 1-2+y = -1+y$ if $1<y<4$

So $$\boxed{V = \pi\int_{y=0}^{y=1} \left((1+\sqrt{y})^2 - (1-\sqrt{y})^2\right) dy + \pi \int_{y=1}^{y=4} \left((1+\sqrt{y})^2 - (-1+y)^2\right) dy}$$

6. (14 pts) (a) (6 pts) Sketch and shade the region bounded between the graph of $y = 1/x$ and the x-axis when $1 \leq x \leq 10$. Then find the exact area of this region.



$$A = \int_1^{10} \frac{1}{x}\,dx = \ln 10 - \ln 1 = \boxed{\ln 10}$$

(b) (6 pts) Consider next the partition of the interval $[1, 10]$ with 9 subintervals of length one, so the partition is $1 < 2 < 3 < \ldots < 9 < 10$. On your sketch from part (a), draw the rectangles corresponding to the *left-end point* Riemann sum associated to this partition. Also, write the expression for this Riemann sum (you can leave this expression as a sum, you do not need to evaluate).

see above for the sketch of the rectangles

$\Delta x = 1$

$$R^{left} = \frac{1}{1}\cdot 1 + \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 1 + \ldots + \frac{1}{9}\cdot 1 = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{9}$$

(c) (2 pts) Is the left-end point Riemann sum from part (b) an over-estimate or an under-estimate of the area in part (a)?

It is an over-estimate of the area in part(a)
(as $y=\frac{1}{x}$ is a decreasing function)

So you proved the inequality $\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{9} > \ln 10$.
This generalizes in an obvious way to give you

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} > \ln n$$

This will be important later.

7. (14 pts) (a) (4 pts) State the Fundamental Theorem of Calculus, both parts.

see notes or textbook

(b) (10 pts) Choose ONE to prove:

(i) FTC part 1 (the one with the derivative of the net-signed area function) – you may assume without proof MVT for integrals, but specify when you are using it;

(ii) FTC part 2 (the one used very often in computations of definite integrals) – you may assume FTC part 1.

see notes or textbook