**Name:** Solution Key **Panther ID:** ____________

**Exam 2** Calculus II Fall 2019

**Important Rules:**

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. Electronic devices (cell phones, calculators of any kind, etc.) should NOT be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
**Violations of any type of this rule will lead to a score of zero on this exam, possibly an automatic grade F for the course and a report for academic misconduct.**

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

**1.** (16 pts) Circle the correct answer (4 pts each):

(a) For the integral $\int \sqrt{9x^2-1}\,dx$ , the following substitution is helpful:

(i) $3x = \sec\theta$ (ii) $w = 9x^2 - 1$ (iii) $w = \sqrt{9x^2-1}$ (iv) $3x = \tan\theta$ (v) $x = 3\sin\theta$

(Don't spend time evaluating the integral. It is not required.)

(b) A parametrization for the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is given by (assume $0 \le t \le 2\pi$)

(i) $x = \sin(2t),\ \ y = \cos(3t)$ (ii)$x = 2t,\ \ y = 3t$ (iii) $x = 2\sin(t),\ \ y = 3\cos(t)$

(iv) $x = 2t,\ \ y = 1 - 3t$ (v) none of the above

(c) Let $T_4$ be the trapezoid approximation with 4 subdivisions of the integral $\int_{-2}^{2}(4-x^2)\,dx$. Then compared with the integral, $T_4$ is an

(i) overestimate (ii) underestimate (iii) exact estimate (iv) cannot tell (more should be known)

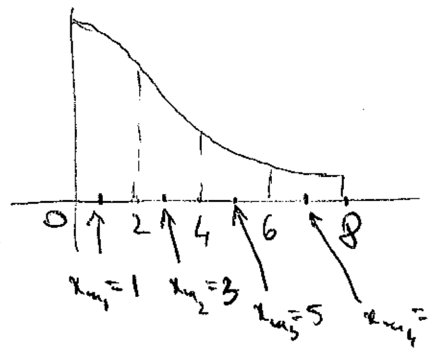
$y = 4 - x^2$ is concave down

(d) Let $S_4$ be the Simpson approximation with 4 subdivisions of the integral $\int_{-2}^{2}(4-x^2)\,dx$. Then compared with the integral, $S_4$ is an

(i) overestimate (ii) underestimate (iii) exact estimate (iv) cannot tell (more should be known)

$y = 4 - x^2$ is a quadratic function
Simpson's method is an exact estimate
for integrals of polynomials of degree $\le 3$.

**2.** (8 pts) Write an expression corresponding to $M_4$, the midpoint approximation with 4 subdivisions, for the integral $\int_0^8 e^{-x^2}\,dx$. Leave your answer in a calculator ready form, you DO NOT need to evaluate.




$$\Delta x = \frac{8-0}{4} = 2$$

$$M_4 = \Delta x\left(f(x_{m_1}) + f(x_{m_2}) + f(x_{m_3}) + f(x_{m_4})\right)$$

$$M_4 = 2\left(e^{-1^2} + e^{-3^2} + e^{-5^2} + e^{-7^2}\right)$$

**3.** (8 pts) For the integral $\displaystyle\int \frac{x+3}{(x+2)^2(x^2+1)}\,dx$

write the partial fraction decomposition and then find the integral, but DO NOT spend time to explicitly find the constants involved. It is NOT required.

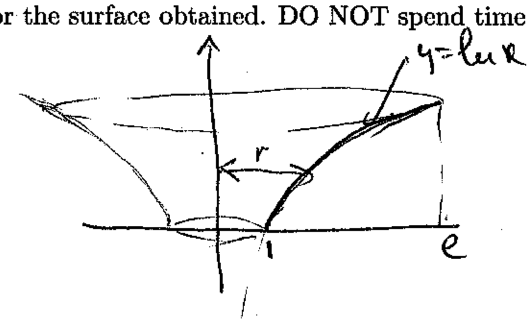
$$\int \frac{x+3}{(x+2)^2(x^2+1)}\,dx = \int\left(\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+1}\right)dx =$$

$$= A\int\frac{1}{x+2}\,dx + B\int\frac{1}{(x+2)^2}\,dx + \int\left(\frac{Cx}{x^2+1} + \frac{D}{x^2+1}\right)dx$$

$$= A\ln|x+2| - B(x+2)^{-1} + C\tfrac{1}{2}\ln(x^2+1) + D\arctan(x) + c$$

$$= A\ln|x+2| - \frac{B}{x+2} + \tfrac{1}{2}C\ln(x^2+1) + D\arctan(x) + c$$

**4.** (8 pts) The curve $y = \ln x$, $1 \le x \le e$ is rotated around the $y$-axis. Set up an integral that gives the surface area for the surface obtained. DO NOT spend time trying to evaluate the integral. It is NOT required.




$$S = \int_?^? 2\pi r\,ds \qquad ds = \sqrt{1+(f'(x))^2}\,dx$$

$$r = x$$

$$\boxed{S = \int_{x=1}^{x=e} 2\pi x\sqrt{1+\left(\frac{1}{x}\right)^2}\,dx}$$

For Problems 5-8, evaluate each integral.

**5.** (12 pts) $\int \arcsin x\, dx$ $= \int \arcsin x \cdot 1 \cdot dx$ $=$

I.B.P. with $du = 1 \cdot dx$ $\quad v = \arcsin x$

$u = x$ $\quad dv = \frac{1}{\sqrt{1-x^2}}\, dx$

$$= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx =$$

substitution

$w = 1-x^2$

$dw = -2x\,dx$

$-\frac{1}{2}\, dw = x\,dx$

$$= x\arcsin x - \int \frac{-\frac{1}{2}dw}{\sqrt{w}} = x\arcsin x + \frac{1}{2}\int w^{-\frac{1}{2}} dw$$

$$= x\arcsin x + \frac{1}{\cancel{2}} \cdot \cancel{2} \cdot w^{\frac{1}{2}} + c =$$

$$= x\arcsin x + (1-x^2)^{\frac{1}{2}} + c = \boxed{x\arcsin x + \sqrt{1-x^2} + c}$$

**6.** (12 pts) $\int_0^{\pi/4} \tan^2 x \sec^4 x \, dx$ $= \int_{x=0}^{x=\frac{\pi}{4}} \tan^2 x \sec^2 x \cdot \sec^2 x \, dx$

$$= \int_{x=0}^{x=\frac{\pi}{4}} \tan^2 x \left(\tan^2 x + 1\right) \cdot \sec^2 x \, dx =$$

sub. $w = \tan x$ $\quad$ $x=0 \Rightarrow w = \tan 0 = 0$

$dw = \sec^2 x \, dx$ $\quad$ $x = \frac{\pi}{4} \Rightarrow w = \tan\left(\frac{\pi}{4}\right) = 1$

$$= \int_{w=0}^{w=1} w^2\left(w^2+1\right) dw = \int_{w=0}^{w=1} \left(w^4 + w^2\right) dw$$

$$= \left(\frac{1}{5}w^5 + \frac{1}{3}w^3\right)\Bigg|_{w=0}^{w=1} = \frac{1}{5} + \frac{1}{3} = \boxed{\frac{8}{15}}$$

**7.** (12 pts) $\displaystyle\int \frac{1}{(9-x^2)^{3/2}}\,dx \quad =$

Trig sub. $\quad x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$

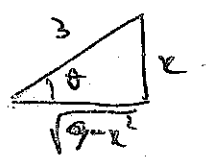
$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta$

so $(9-x^2)^{\frac{3}{2}} = (9\cos^2\theta)^{\frac{3}{2}} = (3\cos\theta)^3 = 27\cos^3\theta$

$$= \int \frac{3\cos\theta\, d\theta}{27\cos^3\theta} = \frac{1}{9}\int \frac{1}{\cos^2\theta}\, d\theta = \frac{1}{9}\int \sec^2\theta\, d\theta$$

$$= \frac{1}{9}\tan\theta + c = \boxed{\frac{1}{9}\frac{x}{\sqrt{9-x^2}} + c}$$

$x = 3\sin\theta \Rightarrow \sin\theta = \frac{x}{3} \Rightarrow \tan\theta = \frac{x}{\sqrt{9-x^2}}$



**8.** (12 pts) $\int \frac{x-2}{x^2-4x+3}\,dx$

Sol. 1 with partial fractions

$$\frac{x-2}{x^2-4x+3} = \frac{x-2}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$

$$x-2 = A(x-3) + B(x-1)$$

... get that $B = \frac{1}{2}$, $A = \frac{1}{2}$

Then $\int \frac{x-2}{x^2-4x+3}\,dx = \int \left(\frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x-3}\right)dx = \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x-3| + c$

Sol. 2 using a substitution

Note that if $w = x^2-4x+3$

$$dw = (2x-4)\,dx = 2\,\underline{(x-2)\,dx}$$

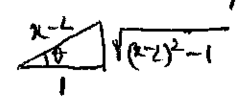
so $\frac{1}{2}dw = (x-2)\,dx$

thus $\int \frac{x-2}{x^2-4x+3}\,dx = \int \frac{\frac{1}{2}dw}{w} = \frac{1}{2}\ln|w| + c = \frac{1}{2}\ln|x^2-4x+3| + c$

Sol. 3 using completion of square and a trig. sub.

$$\frac{x-2}{x^2-4x+3} = \frac{x-2}{x^2-4x+4-1} = \frac{x-2}{(x-2)^2-1}$$

$\sec\theta = x-2 = \frac{x-2}{1}$




$\csc\theta = \frac{x-2}{\sqrt{x^2-4x+3}}$

sub. $x-2 = \sec\theta \implies dx = \sec\theta\tan\theta\,d\theta$

$$\int \frac{x-2}{(x-2)^2-1}\,dx = \int \frac{\sec\theta\cdot\sec\theta\cdot\tan\theta}{\sec^2\theta-1}\,d\theta = \int \frac{\sec^2\theta\cdot\cancel{\tan\theta}}{\tan^{\cancel{2}}\theta}\,d\theta =$$

$$= \cancel{\int \frac{\frac{1}{\cos^2\theta}}{\frac{\sin\theta}{\cos\theta}}\,d\theta} = \int \frac{1+\tan^2\theta}{\tan\theta}\,d\theta = \int\left(\frac{1}{\tan\theta} + \tan\theta\right)d\theta = \int(\cot\theta + \tan\theta)$$

$$= -\ln|\csc\theta| + \ln|\sec\theta| = -\ln\left|\frac{x-2}{\sqrt{x^2-4x+3}}\right| + \ln|x-2| + c$$

**9.** (10 pts) Evaluate the integral $\int_0^{\pi/2} \cos^4 x\, dx$

You could use (without proof) the reduction formula below, or any other method.

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx$$

<u>Sol. 1</u> (using the reduction formula)

$$\int \cos^4 x\, dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\int \cos^2 x\, dx = \frac{\cos^3 x \cdot \sin x}{4} + \frac{3}{4}\left(\frac{\cos x \cdot \sin x}{2} + \frac{1}{2}\int 1\, dx\right)$$

Thus $\int \cos^4 x\, dx = \frac{\cos^3 x \cdot \sin x}{4} + \frac{3}{8}\cos x \cdot \sin x + \frac{3}{8} x + c$

$$\int_0^{\frac{\pi}{2}} \cos^4 x\, dx = \left(\frac{\cos^3 x \cdot \sin x}{4} + \frac{3}{8}\cos x \cdot \sin x + \frac{3}{8} x\right)\Bigg|_{x=0}^{x=\frac{\pi}{2}} = \frac{3}{8}\cdot\frac{\pi}{2} = \frac{3\pi}{16}$$

since $\sin 0 = 0$ & $\cos\frac{\pi}{2} = 0$.

<u>Sol. 2</u> using double angle formulas.

$$\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos(2x)}{2}\right)^2 = \frac{1}{4}\left(1 + 2\cos(2x) + \cos^2(2x)\right) =$$

$$= \frac{1}{4}\left(1 + 2\cos(2x) + \frac{1+\cos(4x)}{2}\right)$$

Thus $\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$ so

$$\int_0^{\frac{\pi}{2}} \cos^4 x\, dx = \int_0^{\frac{\pi}{2}} \left(\frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right) dx =$$

$$= \left(\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right)\Bigg|_{x=0}^{x=\frac{\pi}{2}}$$

$$= \frac{3}{8}\cdot\frac{\pi}{2} = \frac{3\pi}{16}$$

since $\sin(0) = \sin(\pi) = \sin(2\pi) = 0$

hoose ONE. Note the different point values. If you do both, only the larger score will be considered for this blem, but the second score may give some bonus towards a previous problem where your score is smaller.

(A) (12 pts) Prove the reduction formula stated in Problem 9:

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

(B) (8 pts) State and prove the integration by parts formula.

see class notes or textbook