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Exam 3 Calculus II Fall 2019

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (12 pts) The first six terms of a sequence $\{a_n\}_n$ are given below

$$a_1 = \frac{1}{1},\ a_2 = -\frac{3}{2},\ a_3 = \frac{5}{4},\ a_4 = -\frac{7}{8},\ a_5 = \frac{9}{16},\ a_6 = -\frac{11}{32},\ ...$$

(a) (6 pts) Assuming that the pattern continues, find the formula for the general term a_n.

$$a_n = (-1)^{n+1}\frac{2n-1}{2^{n-1}} \quad \text{for } n \geq 1.$$

(b) (6 pts) Using your answer from (a), is the sequence $\{a_n\}_n$ convergent? Briefly justify your answer.

$\{a_n\}_n$ converges to 0, because the denominator grows much faster than the numerator. Even though it oscillates around 0, the oscillation becomes smaller and smaller as $n \to +\infty$

More rigorously: $\lim_{n\to+\infty} |a_n| = \lim_{n\to+\infty} \frac{2n-1}{2^{n-1}} = 0$, so $\lim_{n\to+\infty} a_n = 0$.

2. (12 pts) Evaluate each of the following or show it diverges: (6 pts each)

(a) $\int_0^{+\infty} e^{-3x}\,dx$ $= \lim_{b\to+\infty} \int_0^b e^{-3x}\,dx = \lim_{b\to+\infty}\left(-\frac{1}{3}e^{-3x}\Big|_{x=0}^{x=b}\right) =$

$$= \lim_{b\to+\infty}\left(-\frac{1}{3}e^{-3b} + \frac{1}{3}e^{0}\right) = 0 + \frac{1}{3} = \boxed{\frac{1}{3}}$$

so the improper integral converges to $\frac{1}{3}$.

(b) $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + ...$ $= \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k - \left(\frac{2}{3}\right)^0 =$

geometric with $r = \frac{2}{3} < 1$

$$= \frac{1}{1-\frac{2}{3}} - 1 = 3 - 1 = \boxed{2}$$

so, the above series converges to $\underline{2}$.

3. (10 pts) In each case, answer **True** or **False**. No justification is required for this problem. (2 pts each)

(a) The point of polar coordinates ($r = -2$, $\theta = \frac{9\pi}{4}$) lies in the third quadrant. **True** **False**

(b) Any convergent sequence is bounded. **True** **False**

(c) If $0 \le a_n \le \frac{1}{\sqrt{n}}$ for all $n \ge 1$, then $\sum_{k=1}^{\infty} a_n$ is convergent. **True** **False**

(d) If $0 \le a_n \le \frac{1}{\sqrt{n}}$ for all $n \ge 1$, then $\lim_{n\to\infty} a_n = 0$. **True** **False**

(e) If $\sum_{k=1}^{\infty} |a_k|$ is convergent, then $\sum_{k=1}^{\infty} a_k$ is convergent. **True** **False**

4. (12 pts) Evaluate each of the following or show it diverges: (6 pts each)

(a) $\lim_{k\to+\infty}\left(\frac{1}{k}-\frac{1}{k+2}\right)$ $= 0 - 0 = 0$

(b) $\sum_{k=1}^{+\infty}\left(\frac{1}{k}-\frac{1}{k+2}\right)$ ← observe this is a telescopic series

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+2}\right) = \left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\cdots$$

$$\cdots+\left(\cancel{\frac{1}{n-1}}-\frac{1}{n+1}\right)+\left(\cancel{\frac{1}{n}}-\frac{1}{n+2}\right)$$

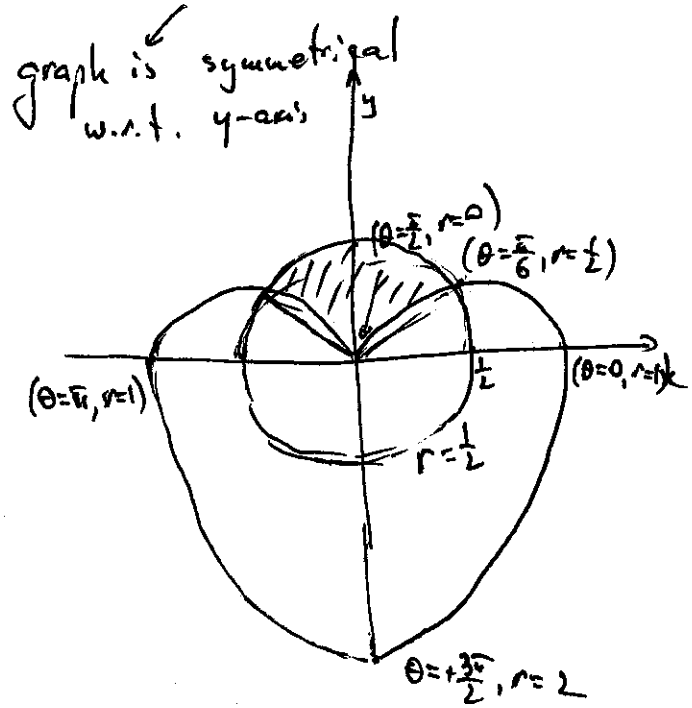
$$S_n = \frac{1}{1}+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$$

so $$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+2}\right) = \lim_{n\to+\infty}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right) = \boxed{\frac{3}{2}}$$

So, the series converges to $\frac{3}{2}$.

5. (14 pts) (a) (6 pts) Sketch in the xy-plane the cardioid $r = 1 - \sin\theta$. Give the polar coordinates of at least four points.

θ	r
0	1
$\frac{\pi}{6}$	$1-\frac{1}{2} = \frac{1}{2}$
$\frac{\pi}{2}$	$1-1 = 0$
π	$1-0 = 1$
$\frac{3\pi}{2}$	$1-(-1) = 2$ ← largest r
2π	1



(b) (8 pts) Set up an expression that gives the area inside the circle $r = 1/2$ but outside of the cardioid $r = 1 - \sin\theta$ (you DO NOT have to evaluate the integral(s) in your expression).

The intersection of the two curves occur when

$\frac{1}{2} = 1 - \sin\theta$ or $\sin\theta = \frac{1}{2}$, so when $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{5\pi}{6}$

$$\text{Shaded Area} = \int_{\theta_1=\frac{\pi}{6}}^{\theta_2=\frac{5\pi}{6}} \left(\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 - \frac{1}{2}(1-\sin\theta)^2 \right) d\theta$$

or, using the symmetry

$$= \cancel{2}\cdot\frac{1}{\cancel{2}} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{4} - (1-\sin\theta)^2 \right) d\theta$$

6. (16 pts) Determine whether each of the following series converges or diverges. Full justification is required.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Apply the integral test: ($f(x) = \frac{1}{x(\ln x)^2}$ is positive and decreasing)

$$\int_2^{+\infty} \frac{1}{x(\ln x)^2}\,dx \underset{\substack{u = \ln x \\ du = \frac{1}{x}dx}}{=} \int_{\ln 2}^{+\infty} \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_{u=\ln 2}^{u=+\infty}$$

$$= 0 - \left(-\frac{1}{\ln 2}\right) = \ln 2$$

As the improper integral converges, the series also converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ is comparable with $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

This is a p-series with $p = \frac{3}{2} > 1$, so it is a convergent series

Direct comparison works as:

$$\frac{\sqrt{n}}{n^2+1} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$$

so $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ is convergent by the Direct comparison Test (and the p-series test)

7. (20 pts) For each of the following series, determine if the series is absolutely convergent (AC), conditionally convergent (CC), or divergent (D). Answer **and carefully** justify your answer. Very little credit will be given just for a guess. Most credit is given for the quality of the justification. (10 pts each)

(a) $\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$ (A.C.) by Ratio Test

$$\rho = \lim_{n \to +\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to +\infty} \frac{\cancel{10^{n+1}}^{\,10}}{\cancel{(n+1)!}_{\,(n+1)}} \cdot \frac{\cancel{n!}}{\cancel{10^n}} = \lim_{n \to +\infty} \frac{10}{n+1} = 0$$

As $\rho = 0 < 1$, the series is absolutely convergent.

Note: Using one of the important Taylor series mentioned in class, you should be able to find the precise value of the series $\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$. Can you do it?

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{2n-1}}$ (C.C.) because:

$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{\sqrt{2n-1}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$ diverges as $\frac{1}{\sqrt{2n-1}} \geq \frac{1}{\sqrt{2n}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{n}}$

so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}} \geq \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} = +\infty$

p-series with $p = \frac{1}{2} < 1$

Thus $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{2n-1}}$ is <u>not</u> absolutely convergent

But $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{2n-1}}$ is convergent by the Alternating Series Test (sequence $\frac{1}{\sqrt{2n-1}}$ is decreasing and $\lim_{n \to +\infty} \frac{1}{\sqrt{2n-1}} = 0$ so A.S.T. applies)

Thus $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{2n-1}}$ is <u>conditionally convergent</u>

8. (12 pts) Choose ONE. If you have time to do both, the second proof may give some bonus towards an earlier problem with a lower score.

(a) State and prove the geometric series theorem.

(b) State and prove the nth-term test for divergence.

See notes on textbook