Name: Solution Key Panther ID: ____________

FINAL EXAM Calculus II Spring 2015

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (10 pts) Evaluate the improper integral or show is divergent $\int_1^{+\infty} \frac{1}{1+x^2}\,dx =$

$$= \lim_{b\to+\infty} \int_1^b \frac{1}{1+x^2}\,dx = \lim_{b\to+\infty} (\arctan b - \arctan 1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$

2. (20 pts) Evaluate the integrals (10 pts each):

(a) $\int x^2 \sin(2x)\,dx \overset{\text{I.B.P.}}{=}$

$du = \sin(2x)dx \qquad v = x^2$

$u = -\frac{1}{2}\cos(2x) \qquad dv = 2x\,dx$

$$= -\frac{1}{2}x^2\cos(2x) + \underbrace{\int \cancel{\tfrac{1}{2}}\cdot\cancel{2}x\cos(2x)\,dx}_{\text{I.B.P. again}} =$$

$du = \cos(2x)dx \qquad v = x$

$u = \frac{1}{2}\sin(2x) \qquad dv = dx$

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \int \frac{1}{2}\sin(2x)\,dx$$

(b) $\int \frac{1}{\sqrt{4+x^2}}\,dx =$

subs. $x = 2\tan\theta$

$\sqrt{4+x^2} = 2\sec\theta$

$dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta}{\cancel{2\sec\theta}}\,d\theta =$$

$$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + c$$

$$= \boxed{\ln\left(\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right) + c}$$

3. (5 pts) Write the partial fraction decomposition. It is NOT required to determine the constants.

$$\frac{1}{(x-2)^3(x+2)(x^2+4)^2} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{(x+2)} + \frac{Ex+F}{(x^2+4)} + \frac{Gx+H}{(x^2+4)^2}$$

4. (15 pts) Circle the correct answer. No justification is necessary for this problem.

(a) Let $s(t)$ be the position of a particle in rectilinear motion during the time interval $a \le t \le b$. The total distance traveled by the particle is given by

(i) $\dfrac{s(b)-s(a)}{b-a}$ (ii) $s(b)$ (iii) $\int_a^b |s'(t)|\,dt$ (iv) $\int_a^b s(t)\,dt$ (v) $s(b)-s(a)$

(b) The expression $\dfrac{d}{dx}\left(\int_0^{x^3} \cos(t^2)\,dt\right)$ is equivalent to

(i) $\sin(x^6)$ (ii) $\cos(x^6)$ (iii) $6x^5\cos(x^6)$ (iv) $3x^2\cos(x^6)$ (v) $3x^2\sin(x^5)$

(c) Let $f(x)$ be a continuous function, positive and concave up on the interval $[a,b]$. Let T_6 be the trapezoidal approximation with 6 subdivisions of the integral $\int_a^b f(x)\,dx$. Then compared with the integral, T_6 is an

(i) overestimate (ii) underestimate (iii) exact estimate (iv) cannot tell (more should be known about f)

(d) The sequence $a_n = 2 + \dfrac{(-1)^n}{n}$, $n \ge 1$ is

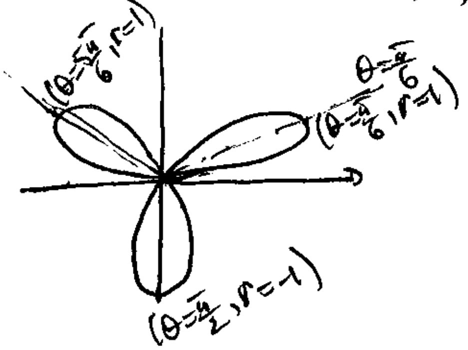
(i) convergent but not monotone (ii) monotone but divergent (iii) bounded but divergent
(iv) eventually decreasing but unbounded (v) none of the above

(e) The average value of the function $f(x)$ over the interval $[a,b]$ is

(i) $f(\frac{a+b}{2})$ (ii) $\dfrac{f(a)+f(b)}{2}$ (iii) $\dfrac{a+b}{2}$ (iv) $\dfrac{f(b)-f(a)}{b-a}$ (v) $\dfrac{1}{b-a}\int_a^b f(x)\,dx$

5. (14 pts) Sketch the rose $r = \sin(3\theta)$ and compute the area of one petal (picture 4pts, computation 10pts).

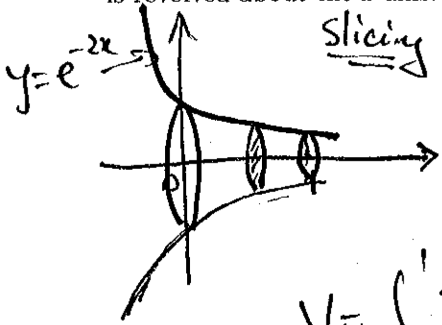
$r=0$ when $3\theta = 0, \pi, 2\pi, 3\pi, \ldots$, so when $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \ldots$



$$A_{1\,petal} = \frac{1}{2}\int_{\theta=0}^{\theta=\frac{\pi}{3}} r^2\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2(3\theta)\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{1-\cos(6\theta)}{2}\,d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{\sin(6\theta)}{6}\right)\Big|_0^{\frac{\pi}{3}} = \boxed{\frac{\pi}{12}}$$

6. (14 pts) Find the volume of the solid that results when the region enclosed by $y = e^{-2x}$, $y = 0$, $x = 0$ and $x = 1$ is revolved about the x-axis. (Computation is required. Sketch of solid is also required.)



Slicing Method

$$V = \int_?^? A_{slice} \cdot Th_{slice}$$

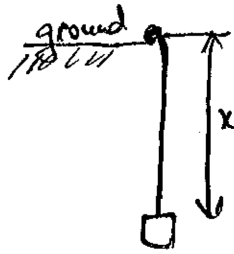
$Th_{slice} = dx$

$A_{slice} = \pi \cdot R^2$, where $R = y = e^{-2x}$

$$V = \int_0^1 \pi (e^{-2x})^2 dx = \pi \int_0^1 e^{-4x} dx = \pi \left(-\frac{1}{4}\right) e^{-4x} \Big|_{x=0}^{x=1}$$

$$V = -\frac{1}{4}\pi \left(e^{-4} - 1\right) = \boxed{\frac{\pi}{4}\left(1 - \frac{1}{e^4}\right)}$$

Pb. 7(a)



The picture shows the moment when there are still x feet of chain + the weight to pull up to the ground. The force necessary at this moment is $F(x) = 0.5x + 100$, so the total work is

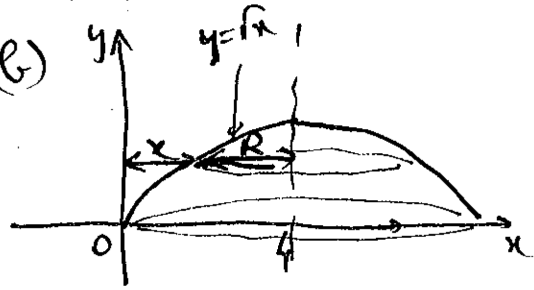
$$\boxed{|W| = \int_?^? F(x) dx = \int_{x=0}^{x=60} (0.5x + 100)\, dx}$$

7. (10 pts) Choose ONE and clearly indicate your choice. Set up the integral only.

(a) A weight of 100 lbs is hanging in a pit 60 feet below ground, suspended (at ground level) by a chain that weighs 0.5 lbs/foot. Set up but **do not evaluate** an integral that gives the the total work to pull the chain and the weight at ground level.

(b) Set up but **do not evaluate** an integral that gives the surface area of the surface generated by the curve $y = \sqrt{x}$, $0 \le x \le 4$ when rotated around the line $x = 4$.

Pb. 7(b)



$$S = \int_?^? 2\pi R \cdot ds$$ ← element of arc length.

$$ds = \sqrt{dx^2 + dy^2} \underset{y=f(x)}{=} \sqrt{dx^2 + (f'(x))^2 dx^2} = \sqrt{1 + (f'(x))^2}\, dx$$

But $f(x) = \sqrt{x}$ so $f'(x) = \frac{1}{2\sqrt{x}}$

$R = 4 - x$

Thus $$\boxed{S = \int_{x=0}^{x=4} 2\pi (4 - x) \sqrt{1 + \frac{1}{4x}}\, dx}$$

8. (14 pts) Determine if the series $\sum_{k=1}^{\infty} \frac{2}{(2k-1)(2k+1)}$ converges. If so, find the sum of the series.

Realize that the series is telescopic

$$\frac{2}{(2k-1)(2k+1)} = \frac{1}{2k-1} - \frac{1}{2k+1}$$ (by guess & check or by partial fractions)

$$S_n = \sum_{k=1}^{n} \frac{2}{(2k-1)(2k+1)} = \sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)$$

So $S_n = 1 - \frac{1}{2n+1}$

Thus $$\sum_{k=1}^{\infty} \frac{2}{(2k-1)(2k+1)} = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(1 - \frac{1}{2n+1}\right) = 1$$

So the series converges to 1.

9. (20 pts) Is the series absolutely convergent, conditionally convergent, or divergent? Justify in each case.

(a) $\sum_{k=2}^{\infty}(-1)^k \sqrt[k]{2}$

Observe that $\lim_{k\to+\infty} \sqrt[k]{2} = \lim_{k\to+\infty} 2^{\frac{1}{k}} = 1$

Thus $\lim_{k\to+\infty} (-1)^k \sqrt[k]{2}$ D.N.E.

(so this limit is $\neq 0$)

Therefore the given series is <u>divergent</u> by the k^{th}-term test.

(b) $\sum_{k=2}^{\infty} \frac{(-1)^k}{(k+1)\sqrt{k}}$

Test absolute convergence

$$\sum_{k=2}^{\infty}\left|\frac{(-1)^k}{(k+1)\sqrt{k}}\right| = \sum_{k=2}^{\infty} \frac{1}{(k+1)\sqrt{k}}$$

But $\frac{1}{(k+1)\sqrt{k}} \leq \frac{1}{k\cdot\sqrt{k}} = \frac{1}{k^{\frac{3}{2}}}$

and $\sum_{k=2}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ is convergent (p-series, with $p = \frac{3}{2} > 1$)

Thus $\sum_{k=2}^{\infty} \frac{1}{(k+1)\sqrt{k}}$ is convergent, by simple comparison test

Thus $\sum_{k=2}^{\infty} \frac{(-1)^k}{(k+1)\sqrt{k}}$ is absolutely convergent

10. (14 pts) Find the radius and the interval of convergence for $\sum_{k=1}^{\infty} \frac{(-1)^k (x-1)^k}{3^k \sqrt[3]{k}}$

Absolute Ratio Test

$$\rho = \lim_{k\to+\infty} \frac{|x-1|^{\cancel{k+1}}}{3^{\cancel{k+1}} \cdot \sqrt[3]{k+1}} \cdot \frac{\cancel{3^k} \cdot \sqrt[3]{k}}{\cancel{|x-1|^k}} = \lim_{k\to+\infty} \left(\frac{|x-1|}{3} \cdot \frac{\sqrt[3]{k}}{\sqrt[3]{k+1}} \right) = \frac{|x-1|}{3}$$

$\rho < 1 \iff \frac{|x-1|}{3} < 1 \iff |x-1| < 3$ Thus radius of convergence is $R = 3$.

$|x-1| < 3 \iff -3 < x-1 < 3 \iff -2 < x < 4$

Test the end-points:

$x = -2$ $\sum_{k=1}^{\infty} \frac{(-1)^k \cdot (-3)^k}{3^k \cdot \sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{(-1)^{2k} \cdot \cancel{3^k}}{\cancel{3^k} \cdot \sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{3}}}$ divergent p-series with $p = \frac{1}{3} <$.

$x = 4$ $\sum_{k=1}^{\infty} \frac{(-1)^k \cdot \cancel{3^k}}{\cancel{3^k} \cdot \sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}}$ convergent by A.S.T

$a_k = \frac{1}{\sqrt[3]{k}} \searrow 0$ so A.S.T applies

Thus, $\boxed{I = (-2, 4]}$

11. (14 pts) (a) (6 pts) Use the Maclaurin series for $\cos x$ to find a numerical series whose sum is $\cos 9°$.

$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, with x in radians

$\cos(9°) = \cos\left(\frac{\pi}{20}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{20}\right)^{2k}$

(b) (8 pts) What is the smallest n so that the partial sum S_n of the series in part (a) approximates $\cos 9°$ with an error less than 10^{-4}? Be sure to carefully justify your answer. You may use that $\pi/20 < 4/20 = 1/5$.

$|R_n(x)| \leq \frac{M \cdot |x - x_0|^{n+1}}{(n+1)!}$. Since the derivatives of $\cos x$ are $\pm \sin x$ or $\pm \cos x$, an upper bound for $|f^{(n+1)}(x)|$ is $M = 1$.

We want $\frac{1 \cdot \left|\frac{\pi}{20} - 0\right|^{n+1}}{(n+1)!} < 10^{-4}$.

Try $n = 3$. $\frac{\left(\frac{\pi}{20}\right)^4}{4!} < \frac{\left(\frac{1}{5}\right)^4}{4!} = \frac{1}{5^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4} < \frac{1}{5^4 \cdot 24} = \frac{1}{10}$

Thus $n = 3$ works, so

$\cos(9°) = \cos\left(\frac{\pi}{20}\right) \approx 1 - \frac{1}{2}\left(\frac{\pi}{20}\right)^2$ is the desired

12. (14 pts) Choose ONE:
(a) State and prove the geometric series theorem.
(b) State and prove the the integration formula for area in polar coordinates. A picture, a sum and a limit should appear in your work.

See notes or textbook