Worksheet week 12 - MAC2312 Fall 2014 NAME: Solution Key

1. True or False. Answer and briefly justify in each case (2pts each).

(a) If $\lim_{k\to+\infty} a_k = 5$ then the series $\sum_{k=1}^{\infty} a_k$ is convergent to 5.

False, series is divergent by k^{th} term divergence test.

(b) If $\sum_{k=1}^{\infty} a_k = 5$ then $\lim_{k\to+\infty} a_k = 5$. False.

Since the series converges, we must have $\lim_{k\to+\infty} a_k = 0$ (again by k^{th} term test)

(c) If $S_n = \sum_{k=1}^{n} a_k$ and $\lim_{n\to\infty} S_n = 5$, then $\sum_{k=1}^{\infty} a_k = 5$. True

$\sum_{k=1}^{\infty} a_k = \lim_{n\to+\infty} S_n = 5$ by the definition of a series.

(d) The series $5-5+5-5+5-5+\dots$ is divergent. True

The sequence of partial sums $S_n = \begin{cases} 5 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$ so $\lim_{n\to+\infty} S_n$ D.N.E.

(e) If $\sum_{k=1}^{\infty} a_k = 5$ and $\sum_{k=1}^{\infty} b_k = 5$ then $\sum_{k=1}^{\infty}(2a_k - b_k) = 5$. True

$\sum_{k=1}^{\infty}(2a_k-b_k) = \lim_{n\to+\infty}\left(\sum_{k=1}^{n}(2a_k-b_k)\right) = \lim_{n\to+\infty}\left(2\sum_{k=1}^{n}a_k - \sum_{k=1}^{n}b_k\right) = 2\lim_{n\to+\infty}\sum_{k=1}^{n}a_k - \lim_{n\to+\infty}\sum_{k=1}^{n}b_k = 2\cdot 5 - 5 = 5$

2. Determine if each of the following series is convergent or divergent. Justify your answer (2.5pts each)

(a) $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty}\frac{1}{k^{\frac{1}{2}}}$

diverges by p-series test

p-series with

$p = \frac{1}{2} < 1$

(b) $\sum_{k=2}^{\infty}\frac{1}{k\ln k} \longleftrightarrow \int_{2}^{+\infty}\frac{1}{x\ln x}\,dx = \dots$

diverges by integral test

$= \ln(\ln x)\Big|_{x=2}^{+\infty} = +\infty$

Note also that $f(x) = \frac{1}{x\ln x}$ is a decreasing positive function when $x\in[2,+\infty)$

(c) $\sum_{k=1}^{\infty}\frac{1}{\sqrt[k]{\pi}} = \sum_{k=1}^{\infty}\frac{1}{\pi^{\frac{1}{k}}}$

$\lim_{k\to+\infty}\frac{1}{\pi^{\frac{1}{k}}} = \frac{1}{\pi^0} = 1 \neq 0$

so series diverges by k^{th} term test.

(d) $\sum_{k=2}^{\infty}\frac{1}{k^2-1}$ convergent

nicest solution is to realize that

$\frac{1}{k^2-1} = \frac{1}{(k-1)(k+1)} = \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$

so the series is telescopic and you can find its actual sum.

Could also use comparison test with $\sum_{k=2}^{\infty}\frac{1}{k^2}$ which is convergent (p-series with $p=2$)

But simple comparison does not work immediately as you don't get the desired inequality. So either use a different coefficient for simple comparison or use limit comparison test