Worksheet week 2 MAC-2312 - Spring 2015 **Name:** Solution Key

1. Consider the sequence:

$$a_1 = \sqrt{3},\ \ a_2 = \sqrt{3 + 2\sqrt{3}},\ \ a_3 = \sqrt{3 + 2\sqrt{3 + 2\sqrt{3}}},\ \ a_4 = \sqrt{3 + 2\sqrt{3 + 2\sqrt{3 + 2\sqrt{3}}}},\ \ldots$$

(a) Find a recursion formula for a_{n+1}.

(b) Use induction to prove that $0 \le a_n \le 3$, for all $n \ge 1$.

(c) Use induction to prove that the sequence $\{a_n\}$ is increasing.

(d) By (b) and (c) it follows that the sequence is convergent (why?). Find its limit.

(a) $a_{n+1} = \sqrt{3 + 2a_n}$ for all $n \geq 1$.

(b) Let $P(n)$ be the statement $0 \le a_n \le 3$ for a given n.

Since $0 \le a_1 = \sqrt{3} \le 3$, the statement $P(1)$ is true ← Basic step

Inductive Step: Assume $P(n)$ true and show that $P(n+1)$ is true.

So we assume $0 \le a_n \le 3$ is true for a given $n \geq 1$

Then $3 + 2\cdot 0 \le 3 + 2a_n \le 3 + 2\cdot 3$ so

$$0 \le \sqrt{3} \le a_{n+1} = \sqrt{3+2a_n} \le \sqrt{9} = 3$$

So we proved the inductive step

By Math Induction, we get that

$0 \le a_n \le 3$ is true for all $n \geq 1$.

(c) Let $P(n)$ be the statement $a_{n+1} > a_n$ for a given n.
We prove that $P(n)$ is true <u>for all</u> $n \geq 1$ by Math Induction

<u>Basic Step</u>: Check that $P(1)$ is true

But $a_2 = \sqrt{3+2\sqrt{3}} > \sqrt{3} = a_1$ so $P(1)$ is indeed true.

<u>Inductive step</u>: Assume $P(n)$ true and show that $P(n+1)$ is true.
So we assume that $a_{n+1} > a_n$ for a given n and we want to show that $a_{n+2} > a_{n+1}$.

But $a_{n+2} = \sqrt{3+2a_{n+1}}$ and $a_{n+1} = \sqrt{3+2a_n}$

As, by assumption, $a_{n+1} > a_n$, then

$$3+2a_{n+1} > 3+2a_n \quad \text{so} \quad \underbrace{\sqrt{3+2a_{n+1}}}_{a_{n+2}} > \underbrace{\sqrt{3+2a_n}}_{a_{n+1}}$$

As we checked both the basic step and the inductive step, it follows that $a_{n+1} > a_n$ is true for al $n \geq 1$.

(d) The sequence $\{a_n\}_n$ is convergent because is monotone (by (c)) and bounded (by (b)).

Let $L = \lim_{n\to+\infty} a_n$. Taking the limit $n \to +\infty$ in the recursive relation $a_{n+1} = \sqrt{3+2a_n}$ we get

$L = \sqrt{3+2L} \Rightarrow$

$\Rightarrow L^2 = 3+2L \Rightarrow L^2 - 2L - 3 = 0 \Rightarrow (L-3)(L+1) = 0$

$\Rightarrow L = 3$ or $L = -1$

But, by (b) $0 \leq a_n \leq 3$, so $0 \leq L \leq 3$

Thus $L=-1$ cannot be the limit. Thus $\boxed{\lim_{n\to+\infty} a_n = 3}$