Exam 1- MAC 2311, F'11 **NAME:** ______________________ **Panther ID:** __________

General Directions: *Read the problems carefully and provide answers exactly to what is requested. Use complete sentences and use notation correctly. Incomprehensible work is worthless. I am grading the work, not just the answer. Don't rush, don't try to do too many steps of a computation at once; work carefully. Good luck!*

1. (25 pts) Find the following limits (if the limit is infinite or does not exist, specify so)

(a) $\lim_{x \to -3} \frac{x^2 - 9}{x^2 + x - 6}$

(b) $\lim_{x \to 2^-} \frac{1 - x}{x - 2}$

Answers: (a) 6/5
Solutions given in class.

(b) $+\infty$

(c) $\lim_{x \to 0} \frac{x + \sin(5x)}{3x + \tan(2x)}$

(d) $\lim_{x \to +\infty} \frac{\cos x}{x}$

Answers: (c) 6/5
Solutions given in class.

(d) 0 use the squeeze theorem.

(e) $\lim_{x \to +\infty} (x - \sqrt{x^2 + 3x})$

Answer: (d) -3/2
Solutions given in class.

2. (10 pts) Let $f(x) = \frac{1}{x}$. Find $f'(x)$ using the limit definition of the derivative.

Solution: Starting from the limit definition of derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} ,$$

we get

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h \to 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

3. (10 pts) Compute the derivative of each of the following functions:

(a) $f(x) = 4x^3 - \frac{\sqrt{x}}{3} + \frac{10}{x^3}$

(b) $f(x) = x^2 \cos x$

Solution: (a) $f(x) = 4x^3 - \frac{1}{3}x^{1/2} + 10x^{-3}$

$$f'(x) = 12x^2 - \frac{1}{3} \cdot \frac{1}{2} x^{-1/2} - 30x^{-4}$$

$$f'(x) = 12x^2 - \frac{1}{6\sqrt{x}} - \frac{30}{x^4}$$

(b) apply product rule .

$$f'(x) = 2x \cos x - x^2 \sin x$$

4. (8 pts) Find $f''(x)$ for $f(x) = \sec x$.

Solution: $f'(x) = \sec x \tan x$. To find $f''(x)$ apply product rule:

$$f''(x) = (\sec x)' \tan x + \sec x (\tan x)' = \sec x \tan x \tan x + \sec x \sec^2 x = \sec x \tan^2 x + \sec^3 x.$$

5. (8 pts) A particle moves on a line so that after t hours it is at $s(t) = 3t^2 + t$ miles from its initial position.

(a) Find the average velocity of the particle in the first two hours. Give units for your answer.

Solution:

$$v_{ave} = \frac{s(2) - s(0)}{2 - 0} = \frac{14 - 0}{2} = 7mph.$$

(b) Find the instantaneous velocity when $t = 2$. Give units for your answer.

Solution: $v(2) = s'(2)$. But $s'(t) = 6t + 1$, so $s'(2) = 13$. Thus $v(2) = 13mph$.

6. (10 pts) Find the equation of the tangent line to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{4}$.

Solution: The point has coordinates (x_0, y_0), where $x_0 = \frac{\pi}{4}$, $y_0 = f(x_0) = \tan(\frac{\pi}{4}) = 1$.
The slope of the tangent line is $m = f'(\frac{\pi}{4})$.
But $f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$. Thus $m = \frac{1}{\cos^2(\frac{\pi}{4})} = \frac{1}{1/2} = 2$.
The tangent line is $y - 1 = 2(x - \frac{\pi}{4})$.

7. (12 pts) Given the function below

$$g(x) = \begin{cases} x^2 + k & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ -kx + 3 & \text{if } x > 1 \end{cases}$$

(a) (6 pts) Find, if possible, a value for the constant k which will make the function $g(x)$ continuous everywhere. If you think there is no such k, justify why not.

Solution: We would like to find k so that

$$\lim x \to 1^- f(x) = \lim x \to 1^+ f(x) = f(1).$$

$$\text{But } \lim x \to 1^- f(x) = 1 + k, \quad \lim x \to 1^+ f(x) = -k + 3, \quad \text{and } f(1) = 3.$$

Thus, k should satisfy at the same time $1 + k = 3$ and $-k + 3 = 3$. There is no value of k that will satisfy both conditions, thus there is no value of k that will make the function continuous.

(b) (6 pts) Sketch the graph of the function $g(x)$ when $k = 2$.
Label carefully the coordinates of important points.

Solution given in class.

8. (12 pts) Sketch a graph of a function $f(x)$ satisfying all of the following conditions.

(i) The function is defined and is continuous everywhere except $x = 0$ and $x = 3$;

(ii) $\lim_{x\to 0^-} f(x) = +\infty$ and $\lim_{x\to 0^+} f(x) = -\infty$;

(iii) $\lim_{x\to 3} f(x) = 1$;

(iv) $\lim_{x\to -\infty} f(x) = -2$ and $\lim_{x\to +\infty} f(x) = 0$.

Solution explained in class.

9. (10 pts) Find k if the curve $y = x^2 + k$ is tangent to the line $y = 6x$.

Solution 1 (with calculus): The slope of $y = 6x$ is 6. Thus we are looking for a point on the graph of $y = x^2 + k$ where the slope is 6. But $y' = 2x$ (k is constant). Thus, solving $2x = 6$, we get $x = 3$. Now for this to be indeed a tangency point the y values of the parabola and the line must coincide.
We get $3^2 + k = 6 \cdot 3$ and solve to obtain $k = 9$.

Solution 2 (without calculus): Finding the intersection point(s) of the parabola and the line means we should solve the system formed by the equations $y = x^2 + k$ and $y = 6x$.
Eliminating y, we get $x^2 + k = 6x$, or $x^2 - 6x + k = 0$.
For the line to be tangent to the parabola, we the above equation should have only one solution. This correspond to finding k so that the left hand-side be a perfect square.
This happens precisely when $k = 3^2 = 9$. The equation becomes $(x - 3)^2 = 0$, so the tangency point is $x = 3$.

10. (10 pts) Let $f(x)$, $g(x)$ be two functions and let $q(x) = \frac{f(x)}{g(x)}$ be the quotient function.

(a) (2 pts) What is $q'(x)$ in terms of f, g and their derivatives? (You are asked to just write the quotient rule.)

Solution:

$$q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

(b) (8 pts) In class we gave a proof for the product rule, but not for the quotient rule. Now you will prove the quotient rule from the product rule. Write $q(x) \cdot g(x) = f(x)$, take the derivative of both sides, solve for $q'(x)$ and show that you will get exactly to the expression form (a).

Solution: Start from $q(x) \cdot g(x) = f(x)$ and take the derivative of both sides. Using product rule in the left side, get

$$q'(x)g(x) + q(x)g'(x) = f'(x).$$

Now solve for $q'(x)$:

$$q'(x)g(x) = f'(x) - q(x)g'(x), ; \text{ so } \quad q'(x) = \frac{f'(x) - q(x)g'(x)}{g(x)}.$$

Now replace $q(x) = \frac{f(x)}{g(x)}$ in the right side and do one more algebra step to show you get to formula in part (a).