Exam 2- MAC 2311, F'11 **NAME:** *Solution Key* **Panther ID:** ______________

General Directions: *Read the problems carefully and provide answers exactly to what is requested. Use complete sentences and use notation correctly. Incomprehensible work is worthless. I am grading the work, not just the answer. Don't rush, don't try to do too many steps of a computation at once; work carefully. Good luck!*

1. (8 pts) Fill in the appropriate words or symbols:

(a) L'Hopital's rule applies directly to indeterminate forms (exceptional cases) of types $\frac{0}{0}$, $\frac{\infty}{\infty}$.

(b) If $f(x)$ is decreasing on $[0, 2]$ then $f(0) > f(1) > f(2)$.

(c) If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is *increasing* on the interval (a, b).

(d) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then the point x_0 is a *relative maximum* for the function $f(x)$.

2. (24 pts) Find the derivative (6 pts each). Simplify your answer when possible.

(a) $f(x) = \ln(4x) + 4^x$

$$f'(x) = \frac{1}{x} + 4^x \ln 4$$

(b) $g(x) = \arcsin(\cos x)$

$$g'(x) = \frac{1}{\sqrt{1 - \cos^2 x}} \cdot (-\sin x) = -1.$$

(c) $h(x) = xe^{-x^2}$

(c) uses product rule and chain rule

$$h'(x) = e^{-x^2} + xe^{-x^2}(-2x)$$

$$h'(x) = (1 - 2x^2)e^{-x^2}$$

(d) $y = x^{\ln x}$

(d) uses logarithmic differentiation

$$\ln y = \ln(x^{\ln x}) = \ln x \cdot \ln x = (\ln x)^2$$

$$\frac{1}{y} y' = 2(\ln x) \frac{1}{x}$$

$$y' = 2\frac{\ln x}{x} x^{\ln x}$$

3. (16 pts) Find each of the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{\ln(x+1)}$ (b) $\lim\limits_{x\to 0^+} (\sin x)^x$

(a) $\dfrac{0}{0}$ case; l'Hopital applies directly.

$$\lim_{x\to 0} \frac{e^{2x}-1}{\ln(x+1)} = \lim_{x\to 0} \frac{2e^{2x}}{\frac{1}{x+1}} = 2$$

(b) 0^0 case; use $A = e^{\ln A}$ trick.

$$\lim_{x\to 0^+} (\sin x)^x = \lim_{x\to 0^+} e^{x\ln(\sin x)} = e^{\lim_{x\to 0^+} x\ln(\sin x)} = e^0 = 1$$

It was shown in class how you get that $\lim\limits_{x\to 0^+} x\ln(\sin x) = 0$.

4. (12 pts) (a) (8 pts) Find the local linear approximation of the function $f(x) = \sqrt{x}$ at $x = 4$.

The local lin. approximation is $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$, where $x_0 = 4$.

$$f(x_0) = f(4) = \sqrt{4} = 2 \quad f'(x) = \frac{1}{2\sqrt{x}},\ f'(x_0) = f'(4) = \frac{1}{4}.$$

Thus $\sqrt{x} \approx 2 + \dfrac{1}{4}(x-4)$, for x near 4.

(b) (4 pts) Use part (a) to approximate $\sqrt{3.98}$ without using a calculator.

$$\sqrt{3.98} \approx 2 + \frac{1}{4}(3.98 - 4) = 1.995$$

5. (12 pts) Find the equation of the tangent line to the curve $3x - x^2y^2 = 2y^3$ at the point $(1,1)$.

Use implicit differentiation.

$3-(2xy^2+x^2 2yy') = 6y^2y'$ then distribute and move the terms containing y' in one side

$3-2xy^2 = (2x^2y+6y^2)y'$, therefore $y' = \dfrac{dy}{dx} = \dfrac{3-2xy^2}{2x^2y+6y^2}$.

The slope of the tangent line is $m = \dfrac{dy}{dx}_{(1,1)} = \dfrac{1}{8}$, so tangent line is $y-1 = \dfrac{1}{8}(x-1)$.

6. (12 pts) A rocket that is launched vertically is tracked by a radar station located on the ground 4 miles from the launch site. What is the vertical speed of the rocket at the instant its distance from the radar station is 5 miles and this distance increases at the rate of 3600 mi/h?

Answer: 6000 mi/h. See class notes for solution.

7. (16 pts) Give a complete graph of the function $f(x) = 3x^4 - 4x^3 + 1$. Your work should include a sign chart for the derivative and the second derivative, the coordinates of the critical and inflection points, the end-behavior of the function. Determine also relative the relative maxima or minima (if any) for the function.

$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$, so critical points are $x = 0, x = 1$.

Do the sign chart and you should see that f' is negative on interval $(-\infty, 0)$, still negative on $(0, 1)$, and positive on $(1, +\infty)$. Thus, $x = 1$ is a relative min, but $x = 0$ is neither a relative min, nor relative max (it is still a stationary point, so the tangent line at $x = 0$ is horizontal as $f'(0) = 0$).
The second derivative is

$f''(x) = 36x^2 - 24x = 12x(3x-2)$, so points where $f''(x) = 0$ are $x = 0, x = \frac{2}{3}$.

Doing the sign chart for $f''(x)$ determine the concavity of f and see that both $x = 0$ and $x = 2/3$ are inflection points.

For the behavior of $f(x)$ at infinity note that $\lim_{x \to \pm\infty} f(x) = +\infty$.

Now you sketch the graph computing also the y-values at critical points for better accuracy.

8. (12 pts) (a) (6 pts) Show that the function $f(x) = \tan x - x$ is increasing over the interval $[0, \pi/2)$.

$$f'(x) = \sec^2 x - 1 = \tan^2 x, \text{ so } f'(x) \geq 0 \text{ for all } x \in [0, \pi/2) .$$

Thus $f(x)$ is increasing on $[0, \pi/2)$.

(b) (6 pts) Use part (a) to show that $\tan x > x$, for any $x \in (0, \pi/2)$.

By part (a), if $x \in (0, \pi/2)$, then $f(x) > f(0)$, because $f(x)$ is increasing. But $f(0) = \tan 0 - 0 = 0$.

$$\text{Thus } \tan x - x > 0, \quad \text{so } \tan x > x \quad \text{for any } x \in (0, \pi/2).$$