Name: Solution Key Panther ID: ____________

Exam 1 - MAC2311 - Summer B 2019

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) These are True or False questions. Circle your answer and briefly justify (3 pts each).

(a) If $\lim_{x\to 3} f(x) = 6$ and $\lim_{x\to 3} g(x) = -1$ then $\lim_{x\to 3}(f(x)+2g(x)) = 4$ **True** **False**

Justification: By properties of limits
$$\lim_{x\to 3}(f(x)+2g(x)) = \lim_{x\to 3} f(x) + 2\cdot\lim_{x\to 3} g(x) = 6 + 2\cdot(-1) = 4$$

(b) The function $f(x) = \frac{\sin x}{\sqrt{x^2+1}}$ is defined and is continuous for all real numbers x. **True** **False**

Justification: $x^2+1>0$ for all x, so $f(x)$ is defined and is continuous for all $x\in\mathbb{R}$.

(c) If $\lim_{x\to a} f(x) = +\infty$ and $\lim_{x\to a} g(x) = +\infty$ then $\lim_{x\to a}\frac{f(x)}{g(x)} = 1$ **True** **False**

Justification: $\frac{\infty}{\infty}$ is an indeterminate form
any result of the limit is possible

(d) The equation $x^3-8x+1=0$ has a real solution in the interval [1,2]. **True** **False**

Justification: Involving I.V.T. is not a correct justification though.
We'll learn a bit later how to use derivatives to get the graph of $f(x) = x^3-8x+1$ and rule out the existence of an x intercept in the interval $[1,2]$.

You could justify the "False" answer with inequalities as follows:

$1\le x\le 2 \Rightarrow 1\le x^2\le 4 \Rightarrow -7\le x^2-8\le -4$. Multiply this by x ($x>0$, as $1\le x\le 2$)

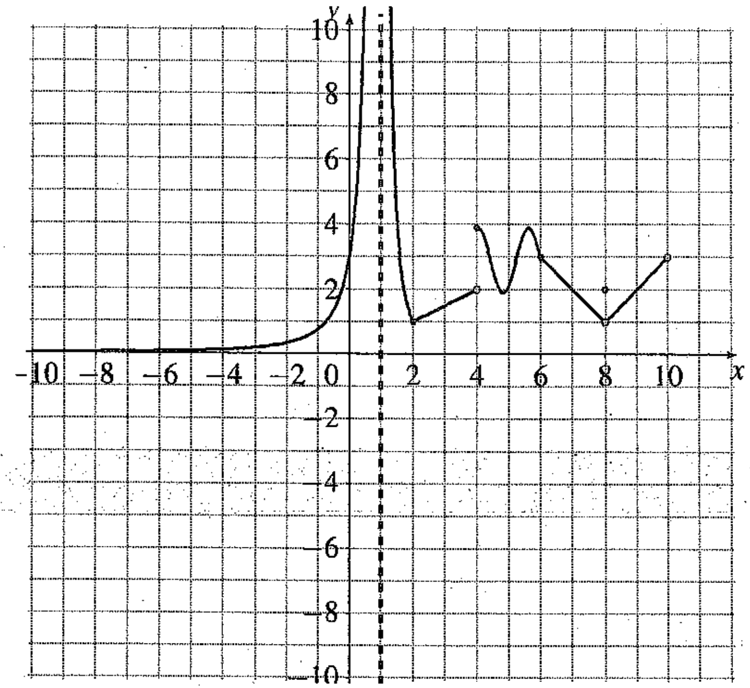
to get $x(x^2-8)\le -4x$
But $x\geq 1$, so $-4x\le -4$
$\Rightarrow x(x^2-8)\le -4x\le -4$ when $1\le x\le 2$

So $x^3-8x\le -4$ or

$x^3-8x+1\le -3<0$ when $1\le x\le 2$

so the equation $x^3-8x+1=0$ cannot have a solution $1\le x\le 2$

Q2. [17 points] Consider the following function $f(x)$:



(a) Find the following limits.

(i) $\lim_{x\to 6} f(x) = 3$

(ii) $\lim_{x\to 8} f(x) = 1$

(iii) $\lim_{x\to 4^+} f(x) = 4$

(iv) $\lim_{x\to 4} f(x)$ D.N.E.

(v) $\lim_{x\to -\infty} f(x) = 0$

(vi) $\lim_{x\to 1} f(x) = +\infty$

(b) What is the domain of this function? ← all possible inputs (x-values) for f

$x \in (-\infty, 1) \cup (1, 10)$

(c) What is the range of the this function? ← all possible outputs (y-values) for f

$y \in (0, +\infty)$

(d) State any asymptotes and their type for this function. If there are none, state that.

$x = 1$ V.A.

$y = 0$ H.A. (when $x \to -\infty$)

(e) List the x-values where this function is **not** continuous.

$x = 1$ (not defined at $x = 1$ and $\lim_{x\to 1} f(x) = +\infty$) and $x = 8$

$x = 4$ (different one sided limits) $\lim_{x\to 8} f(x) = 1 \neq 2 = f(8)$

(f) List the x-values where this function is continuous but **not** differentiable.

$x = 2$, $x = 6$

(g) What is the value of $f'(x)$ at $x = 7$?

$f'(7) = -1$ (slope of tangent line at $x = 7$ is -1)

3. (26 pts) Find the following limits. If the limit is infinite or does not exist, specify so.

a) (3 pts) $\lim_{x \to -4^-} \frac{x}{x+4} =$ $\frac{-4}{0^-} = +\infty$

b) (5 pts) $\lim_{x \to 3} \frac{2x^2 - 18}{x^2 - x - 6} =$ $\frac{0}{0}$ $\lim_{x \to 3} \frac{2(x^2-9)}{(x-3)(x+2)} = \lim_{x \to 3} \frac{2\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x+2)} = \frac{2 \cdot 6}{5} = \boxed{\frac{12}{5}}$

c) (5 pts) $\lim_{x \to -3} \frac{x+3}{\sqrt{7+x}-2} =$ $\frac{0}{0}$ $\lim_{x \to -3} \frac{(x+3)(\sqrt{7+x}+2)}{(\sqrt{7+x}-2)(\sqrt{7+x}+2)} =$

$= \lim_{x \to -3} \frac{(x+3)(\sqrt{7+x}+2)}{7+x-4} = \lim_{x \to -3} \frac{\cancel{(x+3)}(\sqrt{7+x}+2)}{\cancel{x+3}_{\,1}} = \boxed{4}$

d) (3 pts) $\lim_{x \to +\infty} \cos(x) =$ D.N.E. as $\cos x$ oscillates between $-1, 1$.

e) (5 pts) $\lim_{x\to 0}\frac{x\tan(3x)}{\sin^2(5x)} \overset{\frac{0}{0}}{=}$ $\lim_{x\to 0}\frac{x\cdot\frac{\tan(3x)}{3x}\cdot 3x}{\frac{\sin^2(5x)}{(5x)^2}\cdot(5x)^2} =$

$$= \lim_{x\to 0}\frac{\frac{\tan(3x)}{3x}\cdot 3\cancel{x^2}}{\left(\frac{\sin(5x)}{5x}\right)^2\cdot 25\cancel{x^2}} = \boxed{\frac{3}{25}}$$

f) (5 pts) $\lim_{t\to -\infty}\frac{\sqrt{2t^2-t+1}}{5t} \overset{\frac{\infty}{\infty}}{=}$ $\lim_{t\to-\infty}\frac{\sqrt{2t^2}}{5t} = \lim_{t\to-\infty}\frac{\sqrt{2}\cdot|t|}{5t} =$

Junk Rule

$$\overset{t<0}{=} \lim_{t\to-\infty}\frac{\sqrt{2}\cdot(-\cancel{t})}{5\cancel{t}} = \boxed{-\frac{\sqrt{2}}{5}}$$

4. (10 pts) Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = 1/x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{{}^{x}\frac{1}{x+h} - {}^{x+h}\frac{1}{x}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0}\frac{\frac{-h}{x(x+h)}}{\frac{h}{1}} =$$

$$= \lim_{h\to 0}\left(-\frac{\cancel{h}}{x(x+h)}\cdot\frac{1}{\cancel{h}}\right) = \frac{-1}{x^2}$$

(This confirms the power rule $(x^{-1})' = (-1)x^{-2} = -\frac{1}{x^2}$)

5. (12 pts) A stone is thrown straight up from the ground. Its position $s(t)$ in feet above the ground after t seconds is given by $s(t) = 80t - 16t^2$.

(a) (2 pts) When does the stone land back on the ground?

$t = ?$ for $s = 0$

$0 = 80t - 16t^2 \iff 0 = t(80 - 16t)$

$t = 0$ ↑ moment of start $\quad t = \frac{80}{16} = 5s$ ← moment when it lands.

(b) (4 pts) Find the average velocity of the stone in the time interval [0,2] seconds.

$$V_{ave} = \frac{\Delta s}{\Delta t} = \frac{s(2) - s(0)}{2 - 0} = \frac{(80 \cdot 2 - 16 \cdot 2^2) - 0}{2} = 48 \frac{ft}{s}$$

(c) (4 pts) Find the velocity $v(t)$ of the stone at time t .

$$v(t) = s'(t) = (80t - 16t^2)' = 80 - 32t$$

(d) (2 pts) Find the speed with which the stone hits the ground.

Combine (a) with (c)

$v(5) = 80 - 32 \cdot (5) = -90 \frac{ft}{s}$

velocity is negative at 5s as the object moves down

Speed at 5s is $|v(5)| = 90 \frac{ft}{s}$.

6. (12 pts) (a) (2pts) Write the definition for a function *f(x)* to be continuous at *x=a*.

$$\lim_{x \to a} f(x) = f(a)$$

$$\left(\text{or } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a) \right)$$

(b) (5pts) Use this definition to determine whether or not the following function is continuous at *x=0*.

$$f(x) = \begin{cases} \dfrac{x^2+3}{x^2+1} & ,\text{if } x \le 0 \\ \dfrac{\sin(3x)}{x} & ,\text{if } x > 0 \end{cases}$$

$$f(0) = \frac{0^2+3}{0^2+1} = 3$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x^2+3}{x^2+1} = 3$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin(3x)}{3} = 3$$

Thus, by (a) $f(x)$ is continuous at $x=0$.

(e) (5pts) List all asymptotes, vertical or horizontal (if any), of the function *f(x)* from part (b). Justify your answer with limits.

The function has no vertical asymptote

H.A. $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{\sin(3x)}{x} = 0$ (by Squeeze Thm) so $\boxed{y=0}$ is a H.A. as $x \to +\infty$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2+3}{x^2+1} = 1$ so $\boxed{y=1}$ is a H.A. as $x \to -\infty$

7. (10 pts) Find the point(s) on the graph of $f(x) = x^3 - 12x$ where the tangent line is horizontal. (Here you can use the shortcuts for derivative computations.)

Problem can be rephrased as:
find the point(s) where $f'(x) = 0$

$f'(x) = 3x^2 - 12$

$f'(x) = 0 \iff 3x^2 - 12 = 0 \iff 3(x^2-4) = 0$

$3(x-2)(x+2) = 0$

so $\underline{x_1 = 2}$, $\underline{x_2 = -2}$

The points with both coordinates are

$(2, f(2))$ and $(-2, f(-2))$ $\quad (2, -16)$ and $(-2, 16)$

8. (10 pts) Choose ONE of the following:

(a) State and prove the quadratic formula.

(b) Use geometry (areas) to prove the inequality $\sin\theta \leq \theta \leq \tan\theta$ for any $\theta \in (0, \pi/2)$.

See your notes