Name: Solution Key **Panther ID:** ____________

Exam 2 - MAC2311 - Summer B 2019

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (16 points) These are True or False questions. Circle your answer and give a brief justification (4 pts each).

(a) $\ln(A+B) = \ln A + \ln B$, for any A, B. **True** **False**

Justification: For example, if $A=B=1$

$\ln(1+1) = \ln 2 \neq \ln 1 + \ln 1 = 0$

(b) If $y = \sin(5x)$, then $y'' = -25y$. **True** **False**

Justification: $y' = 5\cos(5x)$, $y'' = -25\sin(5x) = -25y$

(c) If $g(x) = x^2 f(x)$, then $g'(x) = 2xf'(x)$. **True** **False**

Justification: $g'(x) = 2x \cdot f(x) + x^2 f'(x)$ (product rule)

(d) If $g(x) = f(x^5)$ and $f'(1) = 2$, then $g'(1) = 10$. **True** **False**

Justification: $g'(x) = f'(x^5) \cdot 5x^4$ (by Chain Rule)

$g'(1) = f'(1) \cdot 5 = 2 \cdot 5 = 10$

2. (35 pts) Compute $\frac{dy}{dx}$ for each of the following functions. Simplify your answer when possible (7 pts each).

a) $y = 3x^5 - 2\sqrt{x} + 10^x$

$y' = 15x^4 - 2 \cdot \frac{1}{2} x^{-\frac{1}{2}} + 10^x \cdot \ln 10 = \boxed{15x^4 - \frac{1}{\sqrt{x}} + 10^x \ln 10}$

b) $y = \ln(\sec x)$

$y' = \frac{1}{\sec x} \cdot \sec x \cdot \tan x$ (by Chain Rule)

$\boxed{y' = \tan x}$

c) $y = e^{3x}\sin(5x)$

$$y' = (e^{3x})'\sin(5x) + e^{3x}\cdot(\sin(5x))'$$

$$y' = 3e^{3x}\sin(5x) + 5e^{3x}\cos(5x)$$

$$\text{or } y' = e^{3x}(3\sin(5x) + 5\cos(5x))$$

d) $y = \cos(\arcsin(\sqrt{x}))$ **+1 bonus point** if you correctly simplify your answer for this one.

$$y' = \sin(\arcsin(\sqrt{x}))\cdot(\arcsin(\sqrt{x}))' = \sin(\arcsin(\sqrt{x}))\cdot\frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot(\sqrt{x})'$$

$$y' = \sin(\arcsin(\sqrt{x}))\cdot\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2}x^{-\frac{1}{2}}$$

$$\text{so } y' = \sqrt{x}\cdot\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} = \boxed{\frac{1}{2\sqrt{1-x}}}$$

(where we used that $\sin(\arcsin\sqrt{x}) = \sqrt{x}$)

e) $y = (1+x^2)^{1/x}$ **Hint:** Logarithmic differentiation

$$\ln y = \ln\left((1+x^2)^{\frac{1}{x}}\right) = \frac{1}{x}\ln(1+x^2) = \frac{\ln(1+x^2)}{x} \quad \Big|\ \text{Take } \frac{d}{dx}$$

$$(\ln y)' = \left(\frac{\ln(1+x^2)}{x}\right)'$$

$$\frac{1}{y}\cdot y' = \frac{\frac{1}{1+x^2}\cdot 2x\cdot x - \ln(1+x^2)\cdot 1}{x^2}$$

$$\text{so } \boxed{y' = (1+x^2)^{\frac{1}{x}}\cdot\frac{\left(\frac{2x^2}{1+x^2} - \ln(1+x^2)\right)}{x^2}}$$

3. (12 pts) (a) (7 pts) Use implicit differentiation to find $\frac{dy}{dx}$ if $2x^3 + 3xy = y^3$.

Take $\frac{d}{dx}$ of both sides of $2x^3 + 3xy = y^3$

$$6x^2 + 3y + 3x \cdot y' = 3y^2 \cdot y'$$

$$6x^2 + 3y = 3y^2 \cdot y' - 3xy'$$

$$\text{so } 3 \cdot y'(y^2 - x) = 3(2x^2 + y)$$

$$\text{so } \frac{dy}{dx} = y' = \frac{2x^2 + y}{y^2 - x}$$

(b) (5 pts) Use part (a) to find the equation of the tangent line to the curve $2x^3 + 3xy = y^3$ at the point $(1, 2)$.

$$m_{tan} = \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{2 \cdot 1^2 + 2}{2^2 - 1} = \frac{4}{3}$$

Tangent line is $\boxed{y - 2 = \frac{4}{3}(x - 1)}$

4. (12 pts) a) (8 pts) Find the local linear approximation of the function $f(x) = x^{1/4}$ at $x_0 = 1$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(1) = \sqrt[4]{1} = 1$$

$$f'(x) = (x^{\frac{1}{4}})' = \frac{1}{4} x^{-\frac{3}{4}} \quad \text{so} \quad f'(1) = \frac{1}{4}$$

Thus $\boxed{\sqrt[4]{x} \approx 1 + \frac{1}{4}(x - 1)}$

b) (4 pts) Use the result of part (a) to approximate $\sqrt[4]{0.92}$.

$$\sqrt[4]{0.92} \approx 1 + \frac{1}{4}(0.92 - 1)$$

or $\sqrt[4]{0.92} \approx 1 + \frac{1}{4}(-0.08) = 1 - 0.02 = 0.98$

Thus $\boxed{\sqrt[4]{0.92} \approx 0.98}$

5. (12 pts) Recall that the volume V of a right circular cylinder is given by $V = \pi r^2 h$, where r is the radius of the base and h is the height. Suppose we have a cylinder for which both dimensions r and h vary with time t.

(a) (6 pts) How are dV/dt, dr/dt and dh/dt related?

$$\boxed{\frac{dV}{dt} = \pi\left(2r\frac{dr}{dt}\cdot h + r^2\frac{dh}{dt}\right)} \quad \text{(product rule)}$$

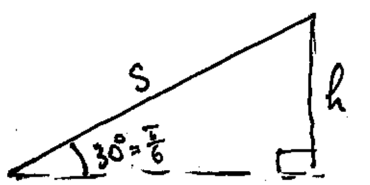
(b) (6 pts) At a certain instant, the height is 8 cm and is increasing at 0.2 cm/s, while the radius is 10 cm and is decreasing at 0.1 cm/s. How fast is the volume changing at that instant? Give units to your answer. Is the volume increasing or decreasing at that instant?

At moment t_0: $h=8$, $\frac{dh}{dt}=0.2$, $r=10$, $\frac{dr}{dt}=-0.1$.

So $$\left.\frac{dV}{dt}\right|_{t=t_0} = \pi\left(2\cdot 10\cdot(-0.1)\cdot 8 + 10^2\cdot(0.2)\right) = \pi(-16+20) = 4\pi\ \frac{cm^3}{s}$$

Thus, at that moment the volume is increasing at a rate of $4\pi\ \frac{cm^3}{s}$.

6. (12 pts) A fighter-jet is climbing at a 30° angle to the horizontal. How fast is the aircraft gaining altitude if its speed is 250 ft/sec ?



$\frac{ds}{dt} = 250\ \frac{ft}{s}$ (speed is the rate of change of distance)

$\frac{dh}{dt} = ?$

$$\frac{1}{2} = \sin(30^\circ) = \frac{h}{s} \Rightarrow$$

$$\Rightarrow h = \frac{s}{2} \quad \Big|\ \text{Take } \frac{d}{dt}$$

$$\frac{dh}{dt} = \frac{1}{2}\frac{ds}{dt}$$

Thus $$\frac{dh}{dt} = \frac{1}{2}\cdot 250 = 125\ \frac{ft}{s}$$

7. (12 pts) Chose ONE of the following proofs. If you have time to do both proofs, the second score may give you some bonus towards a previous problem with a lower score.

(A) Use logarithmic differentiation to prove the power rule formula for derivative.

(B) Find, with proof, the formula for $(\arctan x)'$.

see notes or textbook.