NAME: Answer Key   Panther ID: ____________

Exam 2 - MAC 2313   Spring 2010

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (12 pts) Find the tangent plane of the ellipsoid $2x^2 + 3y^2 + z^2 = 9$ at the point $(1, 1, 2)$.

The ellipsoid is a level surface of the function $F(x,y,z) = 2x^2+3y^2+z^2$.

Thus the normal to the tangent plane is given by

$$\vec{n} = \nabla F(1,1,2)$$

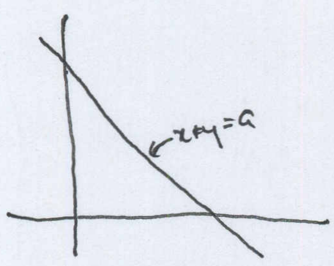
$$\nabla F(x,y,z) = \langle 4x, 6y, 2z \rangle$$

$$\vec{n} = \nabla F(1,1,2) = \langle 4, 6, 4 \rangle$$

Using the point-normal formula, tangent plane is given by: $\boxed{4(x-1)+6(y-1)+4(z-2)=0}$

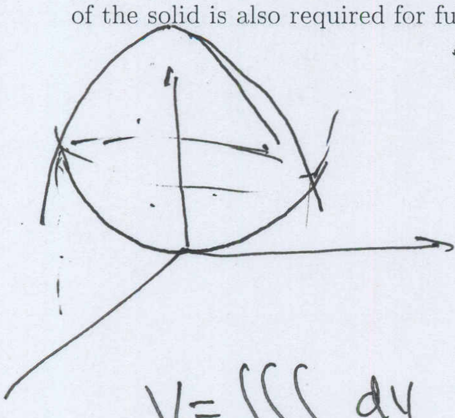
2. (18 pts) Set up iterated integrals to represent each of the following. Do not spend time trying to evaluate the integrals. It is not required.

(a) (9 pts) The mass of the triangular lamina bounded by the coordinate axes and the line $x + y = a$ $(a > 0)$, with density $\rho(x, y) = x + y$.



$$M = \iint_R \delta(x,y)\,dA = \boxed{\int_{x=0}^{x=a}\int_{y=0}^{y=a-x} (x+y)\,dy\,dx}$$

(b) (9 pts) The volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 6 - 2x^2 - 2y^2$. (A rough sketch of the solid is also required for full credit.)



Intersection of the paraboloids

$$\begin{cases} z = x^2+y^2 \\ z = 6-2(x^2+y^2) \end{cases} \Rightarrow z = 6-2z \Rightarrow \begin{cases} x^2+y^2 = 2 \\ z = 2 \end{cases}$$

So the intersection is a circle of radius $\sqrt{2}$ in the plane $z=2$.

$$V = \iiint_G dV = \iint_R \left(\int_{z=x^2+y^2}^{z=6-2(x^2+y^2)} dz \right) dA$$

R is the disk of radius $\sqrt{2}$ so, we'll use polar coords. for R.

$$\boxed{V = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\sqrt{2}}\left(\int_{z=r^2}^{z=6-2r^2} dz\right) r\,dr\,d\theta}$$

3. (15 pts) Find the coordinates of the centroid of the region in the first quadrant that is inside $x^2 + y^2 = (2a)^2$ and outside $x^2 + y^2 = a^2$. Feel free to use anything you can to shorten your work.

Solution with very little computation: Let $C(\bar{x}, \bar{y})$ be the centroid. $\bar{x} = \bar{y}$ by symmetry.

Use Pappus Theorem: Rotate the region around the y-axis

Volume of solid obtained = (Area of Region) $\cdot$ (Dist travelled by centroid)

$$\frac{1}{2}\cdot\frac{4\pi}{3}\left((2a)^3 - a^3\right) = \frac{1}{4}\pi\left((2a)^2 - a^2\right)\cdot 2\pi\bar{x}$$

$$\frac{4}{3}\left(7a^3\right) = \pi\cdot 3a^2\,\bar{x} \implies \bar{x} = \frac{28a}{9\pi}$$

Thus the centroid has coordinates $\left(\frac{28a}{9\pi}, \frac{28a}{9\pi}\right)$

4. (13 pts) Locate and classify all critical points of the function: $f(x, y) = xy + \frac{2}{x} + \frac{4}{y}$.

$$\begin{cases} f_x = y - \frac{2}{x^2} \\ f_y = x - \frac{4}{y^2} \end{cases} \qquad (x,y) \text{ critical pts} \iff \begin{cases} y - \frac{2}{x^2} = 0 \\ x - \frac{4}{y^2} = 0 \end{cases}$$

$$\begin{cases} y = \frac{2}{x^2} \\ x = \frac{4}{y^2} \end{cases} \qquad x = \frac{4}{\left(\frac{2}{x^2}\right)^2} \implies x = x^4 \implies x = 0 \leftarrow f \text{ not defined for } x=0 \text{ or } x = 1$$

Only one critical point at $(1,2)$

Type of critical pt.

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} \frac{4}{x^3} & 1 \\ 1 & \frac{8}{y^3} \end{pmatrix} \quad \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$$

$$D = \det\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} = 4 - 1 = 3 > 0 \implies \boxed{(1,2) \text{ is a local minimum.}}$$

5. (22 pts) The temperature at the point (x, y) of a horizontal plate is given by $T(x, y) = 2y^2 - 4xy - 10x - 2y + 5$ Celsius degrees. Suppose that the y-axis points toward North, the x-axis towards East and that the distances on the plate are measured in meters.

(a) (6 pts) A bug stands at the point $(1, 5)$ and heads directly South. Will it experience an increase or decrease in temperature? At what rate?

Need to compute $(D_{\vec{u}} T)(1,5)$ where $\vec{u} = -\vec{j}$

$(D_{\vec{u}} T)(1,5) = (\nabla T)(1,5) \cdot \vec{u}$

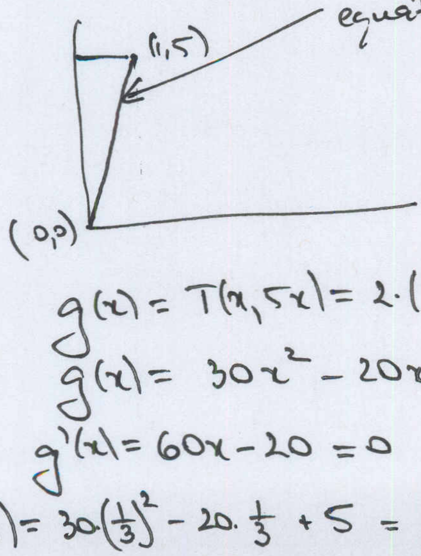
$(\nabla T)(x,y) = \langle -4y - 10,\ 4y - 4x - 2 \rangle$ $\quad (\nabla T)(1,5) = \langle -30, 14 \rangle = -30\vec{i} + 14\vec{j}$

$(D_{\vec{u}} T)(1,5) = (-30\vec{i} + 14\vec{j})(-\vec{j}) = -14$. Temperature decreases at a rate of 14°C/m if it goes south.

(b) (6 pts) If our bug initially stands at the point $(1, 5)$, in which direction should the bug head to experience the greatest rate of increase in temperature? (Give your answer as a vector and as an approximate geographical direction.)

It should go in the direction of the gradient $(\nabla T)(1,5) = -30\vec{i} + 14\vec{j}$ which is a vector in WNW direction

(c) (10 pts) Assume one more time that our bug stands at the point $(1, 5)$, but this time is attracted by a juicy morsel of meat that's exactly at the origin $(0, 0)$. The bug decides to go directly to the morsel on the straight segment between the points $(1, 5)$ and $(0, 0)$. What are the lowest and the highest temperatures that the bug would encounter on this trip? The bug can withstand temperatures in the range of 3° to 50° Celsius, but outside this range even a short time exposure would be fatal. What happens?



equation of the line is $y = 5x$

So we need to consider the function $g(x) = T(x, 5x)$ on the interval $x \in [0,1]$.

$g(x) = T(x, 5x) = 2\cdot(5x)^2 - 4x\cdot(5x) - 10x - 2\cdot(5x) + 5$

$g(x) = 30x^2 - 20x + 5 \quad x \in [0,1]$.

$g'(x) = 60x - 20 = 0 \Rightarrow x = \frac{1}{3}$ critical point

$g(\frac{1}{3}) = 30(\frac{1}{3})^2 - 20\cdot\frac{1}{3} + 5 = \frac{10}{3} - \frac{20}{3} + 5 = \frac{5}{3}$ ← minimum temperature on the path.

$g(0) = 5$

$g(1) = 15$ ← max temperature on the path

Since $\frac{5}{3} = 1.66 < 3$ bug dies

6. (15 pts) Let $g(x,y,z) = f(\rho)$, where $\rho = (x^2+y^2+z^2)^{1/2}$ and f is a differentiable function of a single variable.

(a) Show that $\frac{\partial g}{\partial x} = f'(\rho)\cdot\frac{x}{\rho}$ and write the similar formulas for $\frac{\partial g}{\partial y}$ and $\frac{\partial g}{\partial z}$.

$$\frac{\partial g}{\partial x} = f'(\rho)\cdot\frac{\partial \rho}{\partial x} \quad \text{but} \quad \frac{\partial \rho}{\partial x} = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot\cancel{2}x = \frac{x}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{x}{\rho}$$

Thus $\frac{\partial g}{\partial x} = f'(\rho)\cdot\frac{x}{\rho}$

Similarly $\frac{\partial g}{\partial y} = f'(\rho)\cdot\frac{y}{\rho}$ and $\frac{\partial g}{\partial z} = f'(\rho)\cdot\frac{z}{\rho}$

(b) Use part (a) to show that $||\nabla g||^2 = (f'(\rho))^2$. (As usual, ∇g represents the gradient of g.)

From part (a) $\nabla g = \langle f'(\rho)\cdot\frac{x}{\rho}, f'(\rho)\cdot\frac{y}{\rho}, f'(\rho)\cdot\frac{z}{\rho}\rangle$

$$\nabla g = \frac{f'(\rho)}{\rho}\langle x,y,z\rangle$$

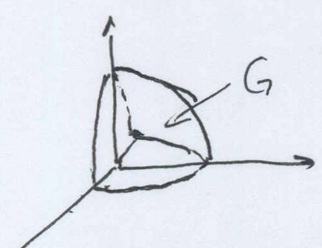
$$\|\nabla g\|^2 = \left(\frac{f'(\rho)}{\rho}\right)^2\cdot\|\langle x,y,z\rangle\|^2 = \frac{(f'(\rho))^2}{\cancel{\rho^2}}\cdot\overset{\cancel{\rho^2}}{(x^2+y^2+z^2)} = (f'(\rho))^2$$

7. (15 pts) Compute the integral by using spherical coordinates:

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dzdydx =$$

$0 \le z \le \sqrt{9-x^2-y^2}$
$0 \le y \le \sqrt{9-x^2}$
$-3 \le x \le 3$

$$= \iiint_G \sqrt{x^2+y^2+z^2}\, dV =$$



$$= \int_{\theta=0}^{\theta=\pi}\int_{\Phi=0}^{\Phi=\frac{\pi}{2}}\int_{\rho=0}^{\rho=3} \rho\cdot\rho^2\sin\Phi\, d\rho\, d\Phi\, d\theta$$

$$= \pi\int_{\Phi=0}^{\Phi=\frac{\pi}{2}}\left(\frac{\rho^4}{4}\Big|_{\rho=0}^{\rho=3}\sin\Phi\right)d\Phi = \frac{81\pi}{4}(-\cos\Phi)\Big|_{\Phi=0}^{\Phi=\frac{\pi}{2}} = \frac{81\pi}{4}$$