NAME: ________________________________ **Panther ID:** ____________

Exam 1 - MAC 2313 Spring 2010

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (15 pts) Circle the correct answer:

(a) The surface $-x^2 + y^2 + 2z^2 = 1$ is a(n)

(i) ellipsoid (ii) hyperboloid (iii) elliptic cylinder (iv) elliptic paraboloid

(b) The surface $x + y^2 + 2z^2 = 1$ is a(n)

(i) ellipsoid (ii) hyperboloid (iii) elliptic cylinder (iv) elliptic paraboloid

(c) The area of the parallelogram with adjacent sides the vectors $\mathbf{u}$ and $\mathbf{v}$ is given by:

(i) $\mathbf{u} \cdot \mathbf{v}$ (ii) $\|\mathbf{u}\| + \|\mathbf{v}\|$ (iii) $\mathbf{u} \times \mathbf{v}$ (iv) $\|\mathbf{u} \times \mathbf{v}\|$

(d) Let $\mathbf{r}(t)$ be a curve in 3-space (in an *arbitrary* parametrization). Denote by $\mathbf{N}(t)$, $\mathbf{B}(t)$ the normal, resp. binormal to the curve. The following two vectors are **always** perpendicular:

(i) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ (ii) $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ (iii) $\mathbf{B}(t)$ and $\mathbf{r}'(t) \times \mathbf{r}''(t)$ (iv) $\mathbf{r}'(t)$ and $\mathbf{N}(t)$

(e) $\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{r}'(t))$ is equivalent to:

(i) $\mathbf{r}(t) \times \mathbf{r}''(t)$ (ii) $\mathbf{r}'(t) + \mathbf{r}''(t)$ (iii) $\mathbf{r}'(t) \times \mathbf{r}''(t)$ (iv) $\|\mathbf{r}'(t)\|^2$

2. (12 pts) Consider the vectors $\mathbf{u} = -2\mathbf{i} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{j} - \mathbf{k}$.

(a) Find the angle between $\mathbf{u}$ and $\mathbf{w}$.

(b) Show that the vector $\mathbf{v}$ is parallel to $\mathbf{u} \times \mathbf{w}$.

(c) Find the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

3. (18 pts) (a) (9 pts) Consider the line L_1 given by $x = 1 - 6t$, $y = 3 + 5t$, $z = 2 + 4t$ and the line L_2 given by the intersection of the planes $x + 2y - z = 2$ and $3x + 2y + 2z = 7$. Show that L_1 and L_2 are parallel.

(b) (9 pts) Find the equation of the plane that contains both lines L_1 and L_2 from part (a).

4. (15 pts) (a) (8 pts) What does the surface $x^2 + z^2 - 6z + 5 = 0$ represent in 3-space?
Draw a sketch of the surface.

(b) (7 pts) A bug is walking on the surface in part (a).
How close to the origin can it get? Explain your answer.

5. (12 pts) Find the arc length of the parametric curve

$$\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}\,, \quad \text{for } 0 \le t \le \pi/2.$$

6. (18 pts) Consider the curve

$$\mathbf{r}(t) = \frac{a}{\sqrt{2}}\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + \frac{a}{\sqrt{2}}\cos t\,\mathbf{k}, \ \text{ where } a \text{ is a constant.}$$

(a) (10 pts) Compute the curvature $\kappa(t)$ of the curve $\mathbf{r}(t)$.

(b) (8 pts) Show that $\mathbf{r}(t)$ represents a circle, by showing that the curve lies on the sphere $x^2 + y^2 + z^2 = a^2$ and on a certain plane. (This should confirm your computation in part (a).)

7. (10 pts) Use the Law of Cosines to prove:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \ , \ \text{ where } \theta \text{ is the angle between } \mathbf{u} \text{ and } \mathbf{v}.$$

8. (10 pts) The purpose of this exercise is to get a formula for the (minimal) distance between two skew lines.

Suppose that L_1 and L_2 are two skew lines in 3-space. Consider P_1, Q_1 two points on L_1 and P_2, Q_2 two points on L_2. Then the minimal distance d between the two lines is given by the magnitude of the projection of the vector $\overrightarrow{P_1P_2}$ onto a vector $\vec{\mathbf{n}}$ that is perpendicular to both $\overrightarrow{P_1Q_1}$ and $\overrightarrow{P_2Q_2}$. (You may assume this fact.)

$$\text{Prove that} \quad d = \frac{|\overrightarrow{P_1P_2} \cdot (\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2})|}{\|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}\|}$$

Note: The above formula gives you the distance between the lines, but does not give you a way to find the points on the two lines that realize this minimal distance.