Name: ______________________ **Panther ID:** ______________________

Exam 2 MTG 3212 Spring 2010

You must show all relevant work to be eligible for credit. Your solutions should be concise and clearly written.

1. (30 pts) (a) Fill in the blanks: The "nine points" referred to in the nine-point circle theorem are:

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(b) Give three different statements equivalent to: "$ABCD$ is a cyclic quadrilateral" (one of them can be the definition).

(c) Define the Appolonius circle associated to two points and a given ratio, and briefly describe how this circle is constructed.

(d) State the Simpson's line Theorem.

(e) Give the definition for the power of a point with respect to a circle and give one equivalent characterization.

(f) Given a line l and two points A, B on the same side of l, how do we find a point P on l that minimizes the sum $|AP| + |PB|$? (No proof is necessary, just the description of how P is found.)

2. (30 pts) Consider two given points in the plane A, B and a given angle α. The goal of this problem is to describe the geometric locus of points P in the plane with the property that $\widehat{APB} = \alpha$.

(a) (5 pts) If $\alpha = \frac{\pi}{2}$, what is the geometric locus? (No proof is needed for this part, just state what the geometric locus is in this case.)

Suppose now that $\alpha < \frac{\pi}{2}$. Do the following construction. Denote by $\mathcal{S}_1$ and $\mathcal{S}_2$ the two half-planes in which the line AB divides the plane. In S_1 determine the point O_1 which has the property that $\widehat{O_1AB} = \widehat{O_1BA} = \frac{\pi}{2} - \alpha$. Draw the circle $\mathcal{C}_1$ centered at O_1 of radius $|O_1A|$ and consider the arc of this circle that is contained in the half-plane $\mathcal{S}_1$ (i.e. $\mathcal{C}_1 \bigcap \mathcal{S}_1$).

(b) (10 pts) Show that any point $P \in \mathcal{C}_1 \bigcap \mathcal{S}_1$ has the property that $\widehat{APB} = \alpha$.

(c) (10 pts) Conversely, show that if $P \in \mathcal{S}_1$ and $P \notin \mathcal{C}_1$, then $\widehat{APB} \neq \alpha$.

(d) (5 pts) In conclusion, what is the geometric locus if $\alpha < \frac{\pi}{2}$?
(You can think at home about the case $\alpha > \frac{\pi}{2}$.)

3. (30 pts) Let $\triangle ABC$ be an acute angle triangle and let D, E, F be the feet of its altitudes (D is on BC, E is on AC, F is on AB).

(a) (10 pts) Find the angles of $\triangle DEF$ in terms of the angles A, B, C of the original triangle.

(b) (10 pts) Prove that $|FE| = |BC| \cos A$. Write without proof similar formulas for $|DF|$ and $|ED|$.

(c) (10 pts) Use parts (a) and (b) to give an alternative proof for the relationship between the circumradius of triangle $\triangle DEF$ and the circumradius of triangle $\triangle ABC$. (*Hint:* Extended Law of Sines.)

4. (30 pts) (a) (15 pts) Given the circle $\mathcal{C}(O,r)$ and two points P, Q in the plane, denote by P', Q' the inverses of P, Q, respectively with respect to the circle $\mathcal{C}(O,r)$.

$$\text{Show that } \ |P'Q'| = r^2 \frac{|PO|}{|OP| \cdot |OQ|} \ .$$

(b) (15 pts) Consider a convex quadrilateral $ABCD$ in the plane and consider a circle $\mathcal{C}(A,r)$ of center A and radius r, where r is an arbitrary positive real number. Consider B', C', D' the inverses of the points B, C, D respectively, with respect to the circle $\mathcal{C}(A,r)$. From the triangle inequality $|B'C'| + |C'D'| \geq |B'D'|$, and part (a) of the problem, deduce the following general inequality that holds for any convex quadrilateral in the plane:

$$|BC| \cdot |AD| + |AB| \cdot |CD| \geq |AC| \cdot |BD| \ .$$

Moreover, show that equality holds if and only if $ABCD$ is a cyclic quadrilateral.