1. Consider a fractional linear transformation

$$L(z) = \frac{az+b}{cz+d}\ , \text{ with } a, b, c, d \in \mathbb{C}.$$

(a) Show that L is a one-to-one map if and only if $ad - bc \neq 0$.

(b) Extend L to a map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, by $L(-\frac{d}{c}) = \infty$ and $L(\infty) = \frac{a}{c}$. Show that if $ad - bc \neq 0$, the extension of L becomes a one-to-one and onto map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

2. Consider a fractional linear transformation

$$L(z) = \frac{az+b}{cz+d}\ , \text{ with } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

(a) Show that if a, b, c, d are **real** numbers then L maps the real axis to the real axis.

(b) Conversely, show that if $L(z)$ maps the real axis to the real axis, then it can be written as $L(z) = \frac{a'z+b'}{c'z+d'}$, with a', b', c', d' **real** numbers and $a'd' - b'c' = 1$.

Hint for (b): Consider two cases $a \neq 0$ and $a = 0$. If $a \neq 0$, then dividing top and bottom by a, the map can be written as

$$L(z) = \frac{z + \tilde{b}}{\tilde{c}z + \tilde{d}}\ .$$

Now prove that if $L(z)$ maps $\mathbb{R}$ to $\mathbb{R}$ then the coefficients $\tilde{b}, \tilde{c}, \tilde{d}$ are real. The determinant 1 condition can be satisfied by one more re-normalization of the coefficients.

In the case $a = 0$, note that b and c must be non-zero, so a similar (and easier) argument can be done.