Name: Solution Key Panther ID: ______________

Exam 2 **MAP 2302: Summer B 2018**

1. (15 pts) These are True/False questions. Answer and give a brief justification (5 pts each).

(a) The UC method can be applied to find a particular solution of $y'' + y = e^x \ln x$ True False

Justification: $f(x) = e^x \cdot \ln x$ is not a UC function

(b) If $y_1(x) = e^x$ and $y_2(x) = e^{2x}$ are solutions of a linear, homogeneous 2nd order ODE with constant coefficients $a_2 y'' + a_1 y' + a_0 y = 0$, then $y_3(x) = e^x \cdot e^{2x} = e^{3x}$ is also a solution. True False

Justification: As y_1, y_2 are lin. independent, the general solution is $y(x) = c_1 e^x + c_2 e^{2x}$, with c_1, c_2 constants; e^{3x} is not a linear combination of e^x, e^{2x}.

(c) The IVP problem $y'' + xy' + x^2 y = 0$, $y(1) = 0$, $y'(1) = 0$, has a unique solution defined on $(-\infty, +\infty)$.

True False

Justification: The existence & uniqueness theorem applies: (coefficients $1, x, x^2$ are continuous on $(-\infty, +\infty)$ and $1 \neq 0$, so we are guaranteed to have a unique solution defined on $(-\infty, +\infty)$.) Actually, the solution is $y(x) \equiv 0$, for $x \in (-\infty, +\infty)$

2. (15 pts) Find the general solution of $y^{(4)} + 3y^{(2)} - 4y = 0$.

characteristic eqn. $\lambda^4 + 3\lambda^2 - 4 = 0$

$(\lambda^2 + 4)(\lambda^2 - 1) = 0$

roots $\lambda_{1,2} = \pm 1$; $\lambda_{3,4} = \pm 2i$

so general solution is

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos(2x) + c_4 \sin(2x)$$

3. (12 pts) Using the UC method, write the form of a particular solution of $y'' - y = te^t$. You DO NOT have to spend time to find the coefficients.

For the homogeneous equation, $\lambda^2 - 1 = 0$, $\lambda = \pm 1$, the general solution is $y_c = c_1 e^t + c_2 e^{-t}$

Initial choice $(At+B)e^t$

so the initial choice should be revised to

$$\boxed{y_p(t) = (At^2 + Bt)e^t}$$

4. (15 pts) Find the general solution of the Cauchy-Euler ODE for $x > 0$: $x^2y'' + 5xy' + 3y = 0$.

Substitution $x = e^t \rightarrow \frac{dx}{dt} = e^t = x$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot x$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx} \cdot x\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot x + \frac{dy}{dx}\,\frac{dx}{dt} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \cdot \frac{dx}{dt} \cdot x + \frac{dy}{dt}.$$

Thus $\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} \cdot x^2 + \frac{dy}{dt}$ so $\frac{d^2y}{dx^2} \cdot x^2 = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

Substituting: get

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 5\frac{dy}{dt} + 3y = 0 \quad \text{or}$$

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 0 \rightarrow \text{charact. equ. } \lambda^2 + 4\lambda + 3 = 0$$

$$\lambda_1 = -1,\ \lambda_2 = -3$$

General solution $y(t) = c_1 e^{-t} + c_2 e^{-3t}$

So the general solution for the original equation is

$$y(x) = c_1 x^{-1} + c_2 x^{-3} \quad \text{for } x > 0$$

Because of issues with units, Pb. 5 is unintentionally quite tricky. The solution below is fine if everywhere in the problems "cm" are replaced by "m". I make these changes and will write separately the solution for the original problem.

5. (18 pts) A spring is such that a force of 10 newtons would stretch it 5 ~~c~~m. The spring hangs vertically and a 2-kg mass is attached to it. After this 2-kg mass comes to rest in its equilibrium position, it is pulled down 3 ~~c~~m below this position and released at $t = 0$ (with zero initial velocity). The medium offers resistance equal to $4x'$, where x' is the velocity in ~~centi~~meters per second.

(a) (6 pts) Set up as an IVP problem.

(b) (10 pts) Solve the IVP to find the displacement function $x(t)$.

(c) (2 pts) Is the motion underdamped (or oscillatory damped), critically damped, or overdamped?

(a) $F = k \cdot x \Rightarrow 10 = k \cdot 5 \Rightarrow k = \frac{10}{5} = 2 \left(\frac{N}{m}\right)$

$m x'' + 4x' + kx = 0$ so

$2x'' + 4x' + 2x = 0$ or

$$\boxed{\begin{aligned} &x'' + 2x' + 1 = 0 \\ &x(0) = x_0 = 3 \\ &x'(0) = 0 \end{aligned}}$$

(b) $\lambda^2 + 2\lambda + 1 = 0$ so $\lambda_{1,2} = -1$

$\overset{\parallel}{(\lambda+1)^2}$

$x(t) = c_1 e^{-t} + c_2 \cdot t e^{-t}$ ← general solution

From $x(0) = 3 \Rightarrow \underline{c_1 = 3}$

$x'(t) = c_1(-1)e^{-t} + c_2 e^{-t} + c_2 \cdot t(-1)e^{-t}$

$0 = x'(0) = c_2 - c_1 \Rightarrow c_2 = c_1 = 3$

Thus $\boxed{x(t) = 3e^{-t} + 3te^{-t}}$

(c) It is a critically damped motion

6. (15 pts) Use the VP method to find the general solution of the differential equation: $y'' + y = \sec^3 x$.
Hint: You can use the formulas given in Problem 7(B) (see next page).

The complementary homog. equ. $y'' + y = 0$ has general solution

$$y_c(x) = c_1 \cos x + c_2 \sin x = c_1 y_1(x) + c_2 y_2(x)$$

We look for a particular solution

$$y_p(x) = c_1(x) y_1(x) + c_2(x) y_2(x) \quad \text{where } y_1(x) = \cos x \quad y_2(x) = \sin x$$

and $c_1'(x)$, $c_2'(x)$ are given by

$$c_1'(x) = -\frac{b(x) y_2}{1 \cdot W_{y_1, y_2}}, \quad c_2'(x) = \frac{b(x) \cdot y_1}{1 \cdot W_{y_1, y_2}} \quad (\text{with } b(x) = \sec^3 x)$$

$$W_{y_1, y_2} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$c_1'(x) = \frac{-\sec^3(x) \cdot \sin x}{1} \qquad c_2'(x) = \frac{\sec^3(x) \cdot \cos x}{1}$$

$$c_1(x) = -\int \sec^3(x) \sin(x)\, dx = \int \frac{\sin x}{\cos^3(x)}\, dx \underset{\substack{w = \cos x \\ dw = -\sin x\, dx}}{=} \int -\frac{1}{w^3}\, dw = \frac{1}{2} w^{-2} = \frac{1}{2\cos^2 x} = \frac{1}{2}\sec^2 x$$

$$c_2(x) = \int \sec^3 x \cdot \cos x\, dx = \int \sec^2 x\, dx = \tan x$$

Thus $y_p(x) = \frac{1}{2}\sec^2 x \cdot (\cos x) + (\tan x) \cdot (\sin x) = \frac{1}{2}\sec x + (\tan x)(\sin x)$

The general solution is

$$y(x) = c_1 \cos x + c_2 \sin x + \frac{1}{2}\sec x + (\tan x)(\sin x)$$

As $y_1(x) = x$ and $y_2(x) = x^2+1$ are independent, the general solution is

$$\boxed{y(x) = c_1 x + c_2(x^2+1)}$$

7. Choose ONE. Note the different point values.

(A) (10 pts) State and prove a theorem on how to get a particular solution of

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b_1(x) + b_2(x)\ ,$$

if you know one particular solution $y_1(x)$ of $a_2(x)y'' + a_1(x)y' + a_0(x)y = b_1(x)$ and another particular solution $y_2(x)$ of $a_2(x)y'' + a_1(x)y' + a_0(x)y = b_2(x)$.

(B) (15 pts) Derive the formulas for $c_1'(x)$ and $c_2'(x)$ from the VP method.
That is, show that if y_1, y_2 are linearly independent solutions of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then a particular solution for $a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$ is given by

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)\ , \quad \text{where}$$

$$c_1'(x) = -\frac{b(x)y_2(x)}{a_2(x)w(x)}\ ,\ c_2'(x) = \frac{b(x)y_1(x)}{a_2(x)w(x)} \quad \text{and } w(x) \text{ denotes the Wronskian of } y_1, y_2.$$

(C) (20 pts) Find the general solution of $(x^2-1)y'' - 2xy' + 2y = 0$.

Solution for (C): Guess one solution: $y_1(x) = x$ is a solution (since $y_1'' \equiv 0$ and $y_1' \equiv 1$)

Then apply reduction of order to find another solution:

Sub: $y = x \cdot v \Rightarrow y' = 1 \cdot v + x \cdot v'$, $y'' = 2v' + x \cdot v''$

Original equation becomes

$$(x^2-1)(2v' + x \cdot v'') - 2x(v + x \cdot v') + 2x \cdot v = 0$$

or $(x^3 - x)v'' + \cancel{2x^2v'} - 2v' - \cancel{2xv} - \cancel{2x^2v'} + \cancel{2xv} = 0$

or $(x^3 - x)w' = 2w$ with $w = v'$

separable

$$\frac{dw}{w} = \frac{2\,dx}{x^3 - x}$$

By partial fractions

$$\frac{2}{x^3-x} = \frac{2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

with some work get $A = -2$, $B = C = 1$

$$\int \frac{dw}{w} = \int \left(-\frac{2}{x} + \frac{1}{x-1} + \frac{1}{x+1}\right)dx$$

$$\ln w = -2\ln x + \ln(x-1) + \ln(x+1) = \ln\left(\frac{x^2-1}{x^2}\right)$$

So $w = \frac{x^2-1}{x^2} = 1 - \frac{1}{x^2} \Rightarrow v(x) = \int\left(1 - \frac{1}{x^2}\right)dx = x + \frac{1}{x} \Rightarrow \underline{y_2(x) = x^2+1}$ ← second soln of the DE