Name: Solution Key Panther ID: ____________

Exam 1 **MAP 2302: Summer B 2018**

1. (18 pts) Label each differential equation with its type - exact, separable, homogeneous (as in section 2.2B), linear, Bernoulli (and not linear), or 'none of the above'. You **DO NOT** have to solve any of them.

(a) $e^y x\,dx + e^x y\,dy = 0$

separable

(b) $\dfrac{dy}{dx} + \dfrac{2y}{x} = \dfrac{x}{y^3}$ Bernoulli (and not linear, $n=-3$)

(c) $(x^2+4xy)\,dx + (2x^2+6y)\,dy = 0$

exact

(d) $(x^2+3y^2)\,dx + (2x^2+y^2)\,dy = 0$

homogeneous

(c) $\dfrac{dy}{dx} + \dfrac{2y}{x} = \cos(y)$

none of the types we learned

(f) $\dfrac{dy}{dx} + \dfrac{2y}{x} = 3x$

linear

2. (10 pts) Every solution of the differential equation $y'' - 9y = 0$ is of the form $y(x) = c_1 e^{3x} + c_2 e^{-3x}$ for some real constants c_1, c_2. Use this to solve the initial value problem $y'' - 9y = 0$, $y(0) = 2$, $y'(0) = 0$.

We just need to find constants c_1, c_2 to satisfy the initial conditions $y(0)=2$ and $y'(0)=0$.

Note that $y'(x) = 3c_1 e^{3x} - 3c_2 e^{-3x}$

So $\begin{cases} 2 = y(0) = c_1 + c_2 \\ 0 = y'(0) = 3c_1 - 3c_2 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 2 \\ c_1 - c_2 = 0 \end{cases} \Rightarrow \dots \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 1 \end{cases}$

So, the solution of the I.V.P is

$$\boxed{y(x) = e^{3x} + e^{-3x}}$$

3. (15 pts) Check that the following differential equation is exact and find its general solution (implicit OK).

$(4x + 3y^2)\,dx + (6xy + 5)\,dy = 0$

$$\frac{\partial M}{\partial y} = 6y = \frac{\partial N}{\partial x}$$ so, it is exact.

$$\frac{\partial F}{\partial x} = 4x+3y^2 \Rightarrow F(x,y) = \int (4x+3y^2)\,dx = 2x^2+3xy^2+g(y)$$

Imposing also $\frac{\partial F}{\partial y} = 6xy+5$ we get

$6xy + g'(y) = 6xy+5$, so $g'(y)=5$ so $g(y) = 5y+c$

The implicit solution of the D.E. is

$F(x,y)=c$, so $\boxed{2x^2+3xy^2+5y=c}$

4. (15 pts) Solve the initial value problem: $2xy\,dx + (x^2+1)\,dy = 0$, with $y(1) = 3$.

Note that the D.E. is separable and exact so you could solve it either way.

Treating it as separable

$$\frac{2x}{x^2+1}\,dx = -\frac{1}{y}\,dy$$

$$\int \frac{2x}{x^2+1}\,dx = -\int \frac{1}{y}\,dy$$

$$\ln(x^2+1) + c = -\ln y$$

(could use y instead of $|y|$ as in the initial condition $y(1)=3>0$)

$$\ln(x^2+1) + \ln y = -c$$

$$\ln(y\cdot(x^2+1)) = -c$$

so $y\cdot(x^2+1) = e^{-c} = \tilde{c}$

From the initial condition $3\cdot 2 = \tilde{c} \Rightarrow \tilde{c} = 6$

so, the explicit solution of the I.V.P is $\boxed{y(x) = \frac{6}{1+x^2}}$

5. (10 pts) Exactly one person in an isolated island population of 10,000 people comes down with a certain disease on a certain day. Suppose the rate at which this disease spreads is proportional to the product of the number of people who have the disease and the people who do not yet have it. Set up an initial value problem that models the spread of this disease. Mention the type of differential equation, but you **do not have to solve it.**

Let $y(t)$ be the number of people that have the disease at time t.

$$\begin{cases} \dfrac{dy}{dt} = k \cdot y\,(10{,}000 - y) \\ y(0) = 1 \end{cases}$$

6. (12 pts) These are True/False questions. Answer and give a brief justification (4 pts each).

(a) The equation $y'' + xy' + 4\sin y = 0$ is a linear, 2nd order, ODE. **True** **False**

Justification: It is not linear because of the $\sin y$ term

(b) The way to approach the differential equation

$\left(x\tan\frac{y}{x} + y\right)dx - x\,dy = 0$ is to use the substitution $y = vx$. **True** **False**

Justification: ~~It is equivalent to~~

Can be rewritten as $\dfrac{dy}{dx} = \tan\left(\dfrac{y}{x}\right) + \dfrac{y}{x}$ so it is homogeneous.

(c) The IVP $\frac{dy}{dx} = x^2 + y^2$, $y(1) = 3$, has a unique solution on an interval $(1-\delta, 1+\delta)$. **True** **False**

Justification:

Since $f(x,y) = x^2 + y^2$ and $\dfrac{\partial f}{\partial y} = 2y$ are continuous everywhere in the plane, the Existence and Uniqueness Theorem applies and guarantees the IVP has a unique solution on some interval around 1.

7. (18 pts) Solve the following Bernoulli differential equation. Check also for any "lost" solution.

$$\frac{dy}{dx} - \frac{2y}{x} = -x^2y^2 \quad \Big| \cdot y^{-2} \qquad \underline{\underline{n=2}}$$

$$y^{-2}\frac{dy}{dx} - 2x^{-1}y^{-1} = -x^2$$

Use the substitution $v = y^{1-n} = y^{1-2} = y^{-1}$

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$$

So $-\frac{dv}{dx} - 2x^{-1}v = -x^2$ or

$$\frac{dv}{dx} + \frac{2}{x}v = x^2 \quad \Big| \cdot \quad \mu = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = \underline{x^2}$$

$$\underline{x^2\frac{dv}{dx} + 2x \cdot v = x^4}$$

$$\frac{d}{dx}(x^2 \cdot v) = x^4 \Rightarrow x^2 \cdot v = \int x^4 dx = \frac{1}{5}x^5 + c$$

So $\frac{1}{y} = v = \frac{1}{5}x^3 + \frac{c}{x^2}$

or $\boxed{y(x) = \frac{5x^2}{x^5 + c}}$

Note also that $\underline{y(x) \equiv 0}$ is also a solution

but it is not ~~in~~ the family of solutions above

C is also in the textbook (see Thm. 2.6 in section 2.4.) but I would be happy if you get this even if we did not cover it.

8. Choose ONE to prove. Note the different point values.

(A) (10 pts) State the formula for $\mu(x)$ used as an integrating factor for $\frac{dy}{dx}+P(x)y=Q(x)$ and then derive (justify) the formula as done in class.

(B) (12 pts) State and prove the theorem showing that a 1st order homogeneous DE (as in section 2.2 B) can be transformed into a separable DE.

(C) (18 pts) Discover a general theorem to determine when a differential equation $M(x,y)dx+N(x,y)dy=0$ has an integrating factor of the type $\mu(y)$.

For A and B see the text or the notes.

(C) We want $\mu(y)M(x,y)\,dx+\mu(y)N(x,y)\,dy=0$ to be exact

We need to have $\frac{\partial}{\partial y}\left(\mu(y)M(x,y)\right)=\frac{\partial}{\partial x}\left(\mu(y)N(x,y)\right)$

or $\mu'(y)\cdot M(x,y)+\mu(y)\frac{\partial M}{\partial y}(x,y)=\mu(y)\frac{\partial N}{\partial x}(x,y)$

or $\mu'(y)\cdot M(x,y)=\mu(y)\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ or

$$\frac{\mu'(y)}{\mu(y)}=\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$$

So the key condition is that the expression on the right side should be just a function of y (be independent of x)

Then the theorem is:

Suppose the differential equation $M(x,y)dx+N(x,y)dy=0$ satisfies the condition that $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ is just a function of y

let's say $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=g(y)$.

Then $\mu(y)=e^{\int g(y)dy}$ is an integrating factor for the D.E.