MAP 2302: Homework – due Monday, July 23 **Name:** Solution Key

1. Find the general power series solution of the differential equation in powers of x (that is, about $x_0 = 0$)

$2y'' + xy' + y = 0$

2. Find the inverse Laplace transform $L^{-1}\left(\frac{1}{s(s^2+4)}\right)$ in two different ways:

(a) using partial fractions (and the table)

(b) using convolution (and the table).

3. Use Laplace transform to solve the following system of linear ODEs

$$\begin{cases} x_1'' + 5x_1 - 2x_2 = 0 \\ x_2'' - 2x_1 + 2x_2 = 0 \end{cases}$$

with initial conditions $x_1(0) = -1,\ x_1'(0) = 0,\ \ x_2(0) = 2, x_2'(0) = 0.$

4. Given that a is a positive constant, use the definition to find the Laplace transform of the step-function

$$u_a(t) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

Note: With this you justified formula (15) from the Laplace transform table (on page 500 textbook).

Pb. 4 is solved in the textbook on the bottom half of page 520.

Solution for Pb. 1

Let $y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

so plugging in the D.E. we get

$$\sum_{n=2}^{\infty} 2n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$4c_2 + c_0 + \sum_{n=1}^{\infty} \left[2(n+2)(n+1) c_{n+2} + (n+1) c_n \right] x^n = 0$$

Thus $\begin{cases} 4c_2 + c_0 = 0 \\ 2(n+2)(n+1) c_{n+2} + (n+1) c_n = 0 & \text{for } n \geq 1 \end{cases}$

$$c_2 = -\frac{1}{4} c_0 = -\frac{1}{2 \cdot 2} c_0$$

$$c_{n+2} = -\frac{1}{2(n+2)} c_n \quad \text{for } n \geq 1$$

$$c_3 = -\frac{1}{2 \cdot 3} c_1$$

$$c_4 = -\frac{1}{2 \cdot 4} c_2 = \frac{1}{(2 \cdot 4) \cdot (2 \cdot 2)} c_0$$

$$c_5 = -\frac{1}{2 \cdot 5} c_3 = \frac{1}{(2 \cdot 5)(2 \cdot 3)} c_1$$

$$c_6 = -\frac{1}{2 \cdot 6} c_4 = -\frac{1}{(2 \cdot 6)(2 \cdot 4)(2 \cdot 2)} c_0$$

$$c_7 = -\frac{1}{2 \cdot 7} c_5 = -\frac{1}{(2 \cdot 7)(2 \cdot 5) \cdot (2 \cdot 3)} c_1$$

Thus,

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x - \frac{1}{2\cdot 2} c_0 x^2 - \frac{1}{2\cdot 3} c_1 x^3 + \frac{1}{(2\cdot 4)(2\cdot 2)} c_0 x^4 + \frac{1}{(2\cdot 5)(2\cdot 3)} c_1 x^5 + \dots$$

so
$$y = c_0\left(1 - \frac{1}{2\cdot 2}x^2 + \frac{1}{(2\cdot 4)(2\cdot 2)}x^4 - \frac{1}{(2\cdot 6)(2\cdot 4)(2\cdot 2)}x^6 + \dots\right) +$$
$$+ c_1\left(x - \frac{1}{2\cdot 3}x^3 + \frac{1}{(2\cdot 5)(2\cdot 3)}x^5 - \frac{1}{(2\cdot 7)(2\cdot 5)(2\cdot 3)}x^7 + \dots\right)$$

Your solution could end here?

Although it is not required in this case you could find a general form for the coefficients, as follows:

$$c_{2k} = (-1)^k \frac{1}{(2\cdot(2k))\cdot(2\cdot(2k-2))\cdot\ldots\cdot(2\cdot 2)} c_0 = \frac{(-1)^k c_0}{2^k\cdot 2^k\cdot k!} = \frac{(-1)^k}{2^{2k}\cdot k!} c_0$$

$$c_{2k+1} = (-1)^k \frac{1}{(2\cdot(2k+1))\cdot(2\cdot(2k-1))\cdot\ldots\cdot(2\cdot 3)} c_1 = \frac{(-1)^k c_1}{2^k (2k+1)(2k-1)\cdot\ldots\cdot 3} =$$

$$= \frac{(-1)^k c_1}{2^k\cdot\frac{(2k+1)(2k)(2k-1)(2k-2)\cdot\ldots\cdot 3\cdot 2}{(2k)(2k-2)\cdot\ldots\cdot 4\cdot 2}} = \frac{(-1)^k c_1}{\cancel{2^k}\cdot\frac{(2k+1)!}{\cancel{2^k}\cdot k!}} = \frac{(-1)^k k!}{(2k+1)!} c_1$$

Thus, the two independent series solutions can be also written as

$$y_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\cdot k!} x^{2k} = \sum_{k=0}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^k}{k!} = e^{-\frac{x^2}{4}} \quad \left(\text{since } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}\right)$$

$$y_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!} x^{2k+1}$$

Let's check that $y_1(x) = e^{-\frac{x^2}{4}}$ is indeed a solution of the D.E.: $2y'' + xy' + y = 0$

$$y_1' = -\frac{1}{2}x\, e^{-\frac{x^2}{4}}$$

$$y_1'' = -\frac{1}{2}e^{-\frac{x^2}{4}} + \frac{1}{4}x^2 e^{-\frac{x^2}{4}}$$

Thus $2y_1'' + x y_1' + y_1 = \cancel{-e^{-\frac{x^2}{4}}} + \cancel{\frac{1}{2}x^2 e^{-\frac{x^2}{4}}} - \cancel{\frac{1}{2}x^2 e^{-\frac{x^2}{4}}} + \cancel{e^{-\frac{x^2}{4}}} = 0$

One can do reduction of order, to obtain a closed form for a second indep. solution of the differential equation.

After some work one gets that a second solution has the form $e^{-\frac{x^2}{4}} \cdot \int e^{\frac{x^2}{4}}\,dx$

This is an odd function (as $e^{\frac{x^2}{4}}$ is even, its antiderivative is odd)

so it must be the $y_2(x)$ whose series we found on the previous page.

Solution(s) for Pb. 2.

(a) Using partial fractions

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$1 = A(s^2+4) + s(Bs+C)$$

Identifying the coefficients we get $\begin{cases} A+B=0 \\ C=0 \\ 4A=1 \end{cases}$ or $\begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \\ C=0 \end{cases}$

so $$\frac{1}{s(s^2+4)} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s}{s^2+2^2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+4)}\right) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{4}\mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right) = \boxed{\frac{1}{4} - \frac{1}{4}\cos(2t)}$$

(b) Using convolution

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+4)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) \overset{\text{commutativity of } *}{=}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) * \mathcal{L}^{-1}\left(\frac{1}{s}\right) = \frac{1}{2}\sin(2t) * 1$$

But $$\left(\frac{1}{2}\sin(2t) * 1\right)(t) = \int_0^t \frac{1}{2}\sin(2\tau)\cdot 1\, d\tau =$$

$$= -\frac{1}{4}\cos(2\tau)\Big|_{\tau=0}^{\tau=t} = \boxed{-\frac{1}{4}\cos(2t) + \frac{1}{4}}$$

Solution for Pb. 3.

Let $X_1 = \mathcal{L}(x_1)$, $X_2 = \mathcal{L}(x_2)$.

After applying Laplace transform to the system, we get.

$$\begin{cases} s^2 X_1 - s x_1(0) - x_1'(0) + 5X_1 - 2X_2 = 0 \\ s^2 X_2 - s x_2(0) - x_2'(0) - 2X_1 + 2X_2 = 0 \end{cases}$$

or, using initial conditions

$$\begin{cases} (s^2+5) X_1 - 2X_2 = -s \\ -2X_1 + (s^2+2)X_2 = 2s \end{cases}$$

Using Cramer's rule

$$X_1(s) = \frac{\begin{vmatrix} -s & -2 \\ 2s & s^2+2 \end{vmatrix}}{\begin{vmatrix} s^2+5 & -2 \\ -2 & s^2+2 \end{vmatrix}} = \frac{-s^3 - 2s + 4s}{s^4 + 7s^2 + 10 - 4} = \frac{-s^3 + 2s}{s^4 + 7s^2 + 6} = \frac{-s(s^2-2)}{(s^2+1)(s^2+6)}$$

$$X_2(s) = \frac{\begin{vmatrix} s^2+5 & -s \\ -2 & 2s \end{vmatrix}}{\begin{vmatrix} s^2+5 & -2 \\ -2 & s^2+2 \end{vmatrix}} = \frac{2s^3 + 10s - 2s}{(s^2+1)(s^2+6)} = \frac{2s(s^2+4)}{(s^2+1)(s^2+6)}$$

Use partial fractions to find $\mathcal{L}^{-1}(X_1(s))$ and $\mathcal{L}^{-1}(X_2(s))$

$$\frac{-s^3+2s}{(s^2+1)(s^2+6)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+6} \quad \Big| \cdot (s^2+1)(s^2+6)$$

$$-s^3 + 2s = (As+B)(s^2+6) + (Cs+D)(s^2+1)$$

Let $s = i$ $\quad -i^3 + 2i = (Ai+B)(i^2+6)$ or $3i = 5Ai + 5B$

so $B = 0$, $A = \frac{3}{5}$

Let $s = i\sqrt{6}$

$$-(i\sqrt{6})^3 + 2i\sqrt{6} = (Ci\sqrt{6} + D)(-6+1)$$

$$8i\sqrt{6} = -5Ci\sqrt{6} - 5D \quad \text{so} \quad D = 0,\ C = -\frac{8}{5}$$

Thus

$$\frac{-s^3+2s}{(s^2+1)(s^2+6)} = \frac{\frac{3}{5}s}{s^2+1} + \frac{-\frac{8}{5}s}{s^2+(\sqrt{6})^2}$$

so $x_1(t) = \mathcal{L}^{-1}\left(\frac{\frac{3}{5}s}{s^2+1} - \frac{\frac{8}{5}s}{s^2+(\sqrt{6})^2}\right) = \frac{3}{5}\cos t - \frac{8}{5}\cos(\sqrt{6}t)$

I'll apply a more ad-hoc method to find the constants for the partial fraction decomposition for $X_2(s)$

$$\frac{2s(s^2+4)}{(s^2+1)(s^2+6)} = \frac{2s(s^2+1+3)}{(s^2+1)(s^2+6)} = 2s\left(\frac{\cancel{s^2+1}}{\cancel{(s^2+1)}(s^2+6)} + \frac{3}{(s^2+1)(s^2+6)}\right) =$$

$$= \frac{2s}{s^2+6} + \frac{6s}{(s^2+1)(s^2+6)} = \frac{2s}{s^2+6} + \frac{6s}{5}\cdot\frac{5}{(s^2+1)(s^2+6)}$$

$$= \frac{2s}{s^2+6} + \frac{6s}{5}\,\frac{[(s^2+6)-(s^2+1)]}{(s^2+1)(s^2+6)} = \frac{2s}{s^2+6} + \frac{6s}{5}\left[\frac{1}{s^2+1} - \frac{1}{s^2+6}\right]$$

Thus $\frac{2s(s^2+4)}{(s^2+1)(s^2+6)} = 2\cdot\frac{s}{s^2+(\sqrt{6})^2} + \frac{6}{5}\,\frac{s}{s^2+1} - \frac{6}{5}\cdot\frac{s}{s^2+(\sqrt{6})^2} =$

$$= \frac{4}{5}\,\frac{s}{s^2+(\sqrt{6})^2} + \frac{6}{5}\,\frac{s}{s^2+1}$$

so $x_2(t) = \mathcal{L}^{-1}\left(\frac{4}{5}\,\frac{s}{s^2+(\sqrt{6})^2} + \frac{6}{5}\,\frac{s}{s^2+1}\right) = \frac{4}{5}\cos(\sqrt{6}t) + \frac{6}{5}\cos t$

So the solution of the system is

$$\begin{cases} x_1(t) = \frac{3}{5}\cos t - \frac{8}{5}\cos(\sqrt{6}t) \\ x_2(t) = \frac{6}{5}\cos t + \frac{4}{5}\cos(\sqrt{6}t) \end{cases}$$