Logistic Differential Equation - general solution

$$\frac{dy}{dt} = ky - \lambda y^2$$, with k, λ positive constants.

Rewrite in the form

$$\frac{dy}{dt} = \lambda y\left(\frac{k}{\lambda} - y\right) \quad \text{or} \quad \underline{\frac{dy}{dt} = \lambda y(M-y) \quad \text{where } M = \frac{k}{\lambda}}$$

Observe that it is a separable diff. equation, so separate the variables

$$\frac{dy}{y(M-y)} = \lambda\, dt$$ and integrate

$$\int \frac{dy}{y(M-y)} = \int \lambda\, dt \qquad (1)$$

For the integral on the left-side, use partial fractions

obtain $$\frac{1}{y(M-y)} = \frac{1}{M}\left(\frac{1}{y} + \frac{1}{M-y}\right)$$ (I used the "guess & adjust" method!)

Thus $$\int \frac{1}{y(M-y)}\,dy = \frac{1}{M}\left(\ln|y| - \ln|M-y|\right)$$ so (1) becomes

$$\frac{1}{M}\left(\ln|y| - \ln|M-y|\right) = \lambda t + c$$ or

$$\ln\left|\frac{y}{M-y}\right| = M\lambda t + Mc \qquad (2)$$

But recall that $M = \frac{k}{\lambda}$, so $M\lambda = k$. Also, we redenote the constant Mc by $\tilde{c}$

With this (2) becomes

$$\ln\left|\frac{y}{M-y}\right| = kt + \tilde{c} \qquad (3)$$

It is convenient to multiply equation (3) by -1 to get

$$-\ln\left|\frac{y}{M-y}\right| = -kt-\tilde{c}, \quad \text{or} \quad \ln\left|\frac{M-y}{y}\right| = -kt-\tilde{c}$$

Exponentiating this we obtain

$$\left|\frac{M-y}{y}\right| = e^{-\tilde{c}}\cdot e^{-kt}$$

Getting rid of absolute value, this leads to

$$\frac{M-y}{y} = Ae^{-kt}, \quad \text{where } A = \pm e^{-\tilde{c}}.$$

With some algebra, one solves for y to get

$$y(t) = \frac{M}{1+A\cdot e^{-kt}} \quad (4),$$ the general family of solutions of the logistic D.E.

The meaning of the constants M and A.

Note that $\lim\limits_{t\to+\infty} y(t) = \lim\limits_{t\to+\infty} \frac{M}{1+Ae^{-kt}} = M$ (since $\lim\limits_{t\to+\infty} e^{-kt} = 0$)

So M is the value to which all solutions converge to in the long run.

In the logistic population growth model, M is called carrying capacity of the environment.

The constant A is determined by the initial condition.

If $y(0)=y_0$, one plugs this in (4) and solve for A to get

$$A = \frac{M-y_0}{y_0}$$

Note that if $0<y_0<M$, $A>0$ and $y(t)$ given by (4) is an increasing function (towards the carrying capacity M)

If $y_0>M$, $A<0$ and $y(t)$ given by (4) is a decreasing function.

If $y_0=M$, $A=0$ and we get the "equilibrium" solution $y(t)\equiv M$, which is a solution of the original D.E.