MAP 2302: Worksheet June 22, 2018 **Group nr:** Solution Key

1. Solve $(3x - y)\,dx - (x + y)\,dy = 0$.

Observe the D.E. is exact since $\frac{\partial}{\partial y}(3x-y) = -1 = \frac{\partial}{\partial x}(-(x+y))$.

(The D.E. is also homogeneous, but it is faster to solve it as an exact D.E.)

We want $F(x,y)$ so that $\frac{\partial F}{\partial x} = 3x-y$ and $\frac{\partial F}{\partial y} = -x-y$

From the first condition

$$F(x,y) = \int (3x-y)\,dx = \frac{3x^2}{2} - xy + g(y).$$

Imposing the second condition we get

$$-\cancel{x} + g'(y) = -x - y, \text{ so } g'(y) = -y, \text{ so } g(y) = -\frac{y^2}{2} + c$$

Thus $F(x,y) = \frac{3x^2}{2} - xy - \frac{y^2}{2} + c$. The general solution of the D.E. is $\boxed{\frac{3x^2}{2} - xy - \frac{y^2}{2} = c}$ ok to leave it in implicit form

2. Solve $(xy + y^2 + x^2)\,dx - x^2\,dy = 0$.

This DE is not exact, but it is homogeneous.

$$\frac{dy}{dx} = \frac{xy+y^2+x^2}{x^2} \iff \frac{dy}{dx} = \frac{xy}{x^2} + \frac{y^2}{x^2} + \frac{x^2}{x^2} \iff \frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1$$

Use the substitution $\frac{y}{x} = v$, or $y = x \cdot v$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ so the D.E. becomes

$$\cancel{v} + x\frac{dv}{dx} = \cancel{v} + v^2 + 1 \iff \frac{dv}{v^2+1} = \frac{dx}{x} \quad \text{separable D.E.}$$

$$\int \frac{dv}{v^2+1} = \int \frac{dx}{x} \Rightarrow \arctan v = \ln|x| + c$$

$$\text{or} \quad v = \tan(\ln|x| + c)$$

Coming back to y, get $\boxed{y = x \cdot \tan(\ln|x| + c)}$

3. Solve the initial value problem:

$$\frac{dy}{dx} - \frac{y}{x} = xe^x\ ,\quad y(1) = e-1\ .$$

It is a linear DE, so use the integrating factor

$$\mu(x) = e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln(x^{-1})} = \frac{1}{x} = x^{-1}$$

Multiply the original D.E. by $\mu = x^{-1}$

$$x^{-1}\frac{dy}{dx} - x^{-2}\cdot y = e^x \quad \text{or} \quad \frac{d}{dx}(x^{-1}\cdot y) = e^x$$

Thus $x^{-1}\cdot y = e^x + c$ or $y(x) = xe^x + cx$.

Using the initial condition $y(1) = e-1$ we get

$$e + c = e - 1 \quad \text{so} \quad c = -1$$

Thus the solution of the IVP is $\boxed{y(x) = xe^x - x}$.

4. Recognize the differential equation $y\ dx + (2x - 5y^3)\ dy = 0$ as one of the types that we studied already (be creative!) and solve it.

Rewrite the D.E. as $\frac{dx}{dy} = -\frac{(2x-5y^3)}{y}$ or

$$\frac{dx}{dy} + \underbrace{\frac{2}{y}}_{P(y)}x = 5y^2$$ and observe it is linear D.E. with unknown $x(y)$.

Integrating factor $\mu = e^{\int P(y)dy} = e^{\int +\frac{2}{y}dy} = e^{+2\ln y} = y^2$

Multiply both sides by μ to get

$$y^2\frac{dx}{dy} + 2y\cdot x = 5y^4 \quad \text{or} \quad \frac{d}{dy}(y^2\cdot x) = 5y^4$$

Thus $y^2\cdot x = \int 5y^4 dy = y^5 + c$, so the solution of the D.E. is $\boxed{x(y) = y^3 + cy^{-2}}$ or just $y^2\cdot x - y^5 = c$ in implicit form.