MAP 2302: Worksheet June 25, 2018 **Group nr:** Solution Key

1. Recognize the type for the DE below and solve it:

$$\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3} \iff \frac{dy}{dx} + \frac{1}{2x} y = x y^{-3}$$

so Bernoulli D.E. with $n = -3$

Multiply both sides by y^3

$$y^3 \frac{dy}{dx} + \frac{1}{2x} y^4 = x \quad (*)$$

and do the substitution $v = y^{1-n} = y^{1-(-3)} = y^4$

Then $\frac{dv}{dx} = 4y^3 \frac{dy}{dx}$ so $\frac{1}{4}\frac{dv}{dx} = y^3 \frac{dy}{dx}$

Substituting in $(*)$, we get

$$\frac{1}{4}\frac{dv}{dx} + \frac{1}{2x} v = x$$, which is a linear D.E.

In standard form $\frac{dv}{dx} + \frac{2}{x} v = 4x$

The integrating factor is $\mu = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$,

so we multiply by μ

$$x^2 \frac{dv}{dx} + 2xv = 4x^3$$

or $\frac{d}{dx}(x^2 \cdot v) = 4x^3$

Thus, we get $x^2 \cdot v = \int 4x^3 dx = x^4 + c$

or $v = x^2 + cx^{-2}$

Since $y^4 = v = x^2 + cx^{-2} \Rightarrow \boxed{y = \left(x^2 + \frac{c}{x^2}\right)^{\frac{1}{4}}}$

2. (a) Check that the DE $y'' + 4y = 0$ has a family of solutions of the form $y(x) = c_1 \sin(2x) + c_2 \cos(2x)$.

For parts (b) and (c), take for granted that this is the **general** solution of the DE (we'll see this in Chapter 4).

If $y(x) = c_1 \sin(2x) + c_2 \cos(2x)$

$y'(x) = 2c_1 \cos(2x) - 2c_2 \sin(2x)$

$y''(x) = -4c_1 \sin(2x) - 4c_2 \cos(2x) = -4(c_1 \sin(2x) + c_2 \cos(2x)) = -4y(x)$

Thus $y(x) = c_1 \sin(2x) + c_2 \cos(2x)$ are solutions for the DE. $y'' + 4y = 0$

(b) Find a solution for the DE in part (a) satisfying the initial value condition $y(0) = 1$, $y'(0) = 4$, or, if this is not possible, explain.

We find the constants c_1, c_2 to satisfy the initial condition.

$$\begin{cases} 1 = y(0) = c_1 \sin(0) + c_2 \cos(0) \\ 4 = y'(0) = 2c_1 \cos(0) - 2c_2 \sin(0) \end{cases} \Rightarrow \begin{cases} c_2 = 1 \\ 2c_1 = 4 \end{cases} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = 1 \end{cases}$$

Thus $y(x) = 2\sin(2x) + \cos(2x)$ ~~satisfies~~ solves the I.V.P.

(c) Find a solution for the DE in part (a) satisfying the boundary value condition $y(0) = 1$, $y'(\pi/4) = 4$, or, if this is not possible, explain.

We want to find c_1, c_2 to satisfy

$$\begin{cases} 1 = y(0) = c_1 \sin(0) + c_2 \cos(0) \\ 4 = y'(\frac{\pi}{2}) = 2c_1 \cos(2 \cdot \frac{\pi}{4}) - 2c_2 \sin(2 \cdot \frac{\pi}{4}) \end{cases} \Rightarrow \begin{cases} c_2 = 1 \\ -2c_2 = 4 \end{cases} \quad \text{contradiction}$$

So, there is no solution to this B.V.P.

3. (a) Consider the IVP

$$\frac{dy}{dx} = x\cos^2 y\ , \quad y(0) = \frac{\pi}{2}\ .$$

Does the Fundamental Theorem for 1st order ODE apply to guarantee the existence and the uniqueness of the solution for the above IVP? Briefly explain.

$f(x,y) = x\cos^2 y$

$\frac{\partial f}{\partial y}(x,y) = 2x\cos y(-\sin y)$

Both f and $\frac{\partial f}{\partial y}$ are continuous for all (x,y) in the plane, so the assumptions of the theorem are satisfied.

Thus, we are guaranteed the existence & uniqueness of a solution defined on an interval containing 0.

(b) Now observe that the DE is separable and try to solve the IVP. Can you explain the apparent contradiction between parts (a) and (b)?

We separate the variables

$$\frac{dy}{\cos^2 y} = x\,dx \quad \text{or} \quad \sec^2 y\,dy = x\,dx$$

We integrate $\quad \int \sec^2 y\,dy = \int x\,dx$

so $\tan y = \frac{1}{2}x^2 + c \leftarrow$ implicit form of the solution

We ~~impose~~ try to impose the initial condition to find c.

$\tan\frac{\pi}{2} = \frac{1}{2}\cdot 0 + c$. But $\tan\frac{\pi}{2}$ is <u>not defined</u>!

What is going on?

Note that in the separation of variables, we "lost" the solution $y(x) \equiv \frac{\pi}{2} \leftarrow$ constant

Note that this is indeed a solution and satisfies the initial condition.

So this is the solution of the IVP guaranteed by the theorem.