MAP 2302: Worksheet July 9, 2018 **Group nr:** Solution key

1. Decide if the UC Method for finding a particular solution applies for the DEs below and, if it does, write the form of your particular solution. You **DO NOT** have to find the coefficients:

(a) $y^{(4)} - y^{(2)} - 12y = x^2$ $b(x) = x^2$ so the initial choice for y_p is $y_p(x) = Ax^2 + Bx + C$

The characteristic equ. for the homogeneous equation is
$\lambda^4 - \lambda^2 - 12 = 0 \iff (\lambda^2 - 4)(\lambda^2 + 3) = 0$, so roots are $\lambda_{1,2} = \pm 2$, $\lambda_{3,4} = \pm i\sqrt{3}$

The complementary function is $y_c = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos(\sqrt{3}x) + c_4 \sin(\sqrt{3}x)$.

No "interference" with the initial choice of y_p. So we'll choose

$$\underline{y_p(x) = Ax^2 + Bx + C}$$

(b) $(\sin x)y'' + (\cos x)y = e^x$

UC Method does not apply since the coefficients of the homogeneous equation (the a's) are <u>not</u> constant.

(c) $y'' - 4y' + 5y = xe^{2x} \sin x$

The initial choice for y_p: $y_p = (Ax + B)e^{2x}(C\sin x + D\cos x)$

or $y_p(x) = (ax + b)e^{2x}\cos x + (cx + d)e^{2x}\sin x$

But we look at the complementary function

$\lambda^2 - 4\lambda + 5 = 0$ charact. equation

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

so $y_c = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x$

So now there is "interference" between our initial choice for y_p and the complementary function

(The terms $b \cdot e^{2x} \cos x$ and $d e^{2x} \sin x$ from initial choice of y_p satisfy the homogeneous equation.)

Thus we adjust the initial choice of y_p to:

$$y_p = (ax^2 + bx)e^{2x}\cos x + (cx^2 + dx)e^{2x}\sin x$$

with constants a, b, c, d to be determined

Equivalent form would be: $y_p = (Ax^2 + Bx)e^{2x}(C\sin x + D\cos x)$

2. Use the VP method to find the general solution of the DE

$$y'' + 6y' + 9y = \frac{e^{-3x}}{x^3}$$

The complementary equation $y'' + 6y' + 9y = 0$ has characteristic equation $\lambda^2 + 6\lambda + 9 = 0$ with roots $\lambda_1 = \lambda_2 = -3$.

Thus the complementary function is

$$y_c = c_1 e^{-3x} + c_2 x \cdot e^{-3x}$$

Apply the V.P. method to look for a particular sol'n of the non-homog. eq'n of the form $y_p = c_1(x) y_1 + c_2(x) y_2$, where $y_1 = e^{-3x}$, $y_2 = x \cdot e^{-3x}$

From the theory we know that if

$$c_1'(x) = -\frac{b \cdot y_2}{a_2 \cdot W_{y_1, y_2}}, \quad c_2'(x) = \frac{b \cdot y_1}{a_2 \cdot W_{y_1, y_2}}$$

y_p of the form above will be a solution.

Here $a_2 \equiv 1$ (coef. of y''), $b = \frac{e^{-3x}}{x^3}$, $y_2 = x \cdot e^{-3x}$, $y_1 = e^{-3x}$

wronskian $\to$ $W_{y_1, y_2} = \begin{vmatrix} e^{-3x} & x \cdot e^{-3x} \\ -3e^{-3x} & e^{-3x}(-3x+1) \end{vmatrix} = e^{-6x}[\cancel{-3x} + 1 \cancel{+3x}] = e^{-6x}$

Replacing $c_1'(x) = \frac{-\frac{\cancel{e^{-3x}}}{x^3} \cdot x \cancel{e^{-3x}}}{\cancel{e^{-6x}}} = -\frac{1}{x^2} \Rightarrow c_1(x) = \int -\frac{1}{x^2} dx = \frac{1}{x}$

$$c_2'(x) = \frac{\frac{\cancel{e^{-3x}}}{x^3} \cdot \cancel{e^{-3x}}}{\cancel{e^{-6x}}} = \frac{1}{x^3} \Rightarrow c_2(x) = \int \frac{1}{x^3} dx = -\frac{1}{2} x^{-2} = -\frac{1}{2x^2}$$

Thus $y_p(x) = \frac{1}{x} \cdot e^{-3x} - \frac{1}{2x^{\cancel{2}}} \cdot \cancel{x} e^{-3x} = \left(\frac{1}{x} - \frac{1}{2x}\right) e^{-3x} = \frac{1}{2x} e^{-3x}$

The general solution of the D.E. is:

$$\boxed{y(x) = c_1 e^{-3x} + c_2 x \cdot e^{-3x} + \frac{1}{2x} e^{-3x}}$$

3. Solve the following Cauchy-Euler equation

$x^3y'''-3x^2y''+6xy'-6y=0$

Transformation $x=e^t$ (Note that $\underline{\frac{dx}{dt}=(e^t)'=e^t=x}$)

By Chain Rule $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=\frac{dy}{dx}\cdot x$. Thus $\boxed{x\cdot\frac{dy}{dx}=\frac{dy}{dt}}$ (1)

Taking one more derivative ~~w.r.t.~~ with respect to t of both sides of (1)

one gets $\boxed{x^2\cdot\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}}$ (2) ← was done in class, will not write it again here.

To get the way to replace $x^3\cdot\frac{d^3y}{dx^3}$ take one more derivative in t of both sides of (2).

$$\frac{d}{dt}\left(x^2\cdot\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}$$

$$\Leftrightarrow\quad 2x\frac{dx}{dt}\cdot\frac{d^2y}{dx^2}+x^2\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}$$ (we applied product rule for the left side)

Now, by chain Rule $\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)\cdot\frac{dx}{dt}$, thus the equality above ~~left side~~ becomes:

$$2x^2\cdot\frac{d^2y}{dx^2}+x^2\cdot\frac{d^3y}{dx^3}\cdot x=\frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}$$

So we get $$x^3\cdot\frac{d^3y}{dx^3}=\frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}-2x^2\frac{d^2y}{dx^2}$$

Using (2) to replace $x^2\cdot\frac{d^2y}{dx^2}$ we get

$$x^3\cdot\frac{d^3y}{dx^3}=\frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}-2\frac{d^2y}{dt^2}+2\frac{dy}{dt}$$ or

$$\boxed{x^3\cdot\frac{d^3y}{dx^3}=\frac{d^3y}{dt^3}-3\frac{d^2y}{dt^2}+2\frac{dy}{dt}}\quad(3)$$

Continuation Pb. 3

We use $\boxed{x \cdot \frac{dy}{dx} = \frac{dy}{dt}}$ (1) $\boxed{x^2 \cdot \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}}$ (2) $\boxed{x^3 \cdot \frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}}$ (3)

and replace in the original equation to get

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 6\frac{dy}{dt} - 6y = 0$$

or $$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = 0$$

Characteristic equation of this is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

whose roots are $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$

The general solution of the transformed equation is

$$y(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{3t}$$

but doing the transformation $x = e^t$ back (or $t = \ln x$)

get the general solution of the original C-E equ.

$$\boxed{y(x) = c_1 x + c_2 x^2 + c_3 x^3}$$