Midterm - MAD 2104 - Summer A 2014 **NAME:** Solution Key

To receive credit you MUST SHOW ALL YOUR WORK.

1. (10 pts) Show that $(p \to r) \wedge (q \to r)$ is logically equivalent with $(p \vee q) \to r$.

Can be done with truth table, but here is the solution with logical equivalences. Key is the equivalence $a \to b \equiv \neg a \vee b$. (*)

$$(p \to r) \wedge (q \to r) \underset{(*)}{\equiv} (\neg p \vee r) \wedge (\neg q \vee r) \underset{\text{distrib. property}}{\equiv} (\neg p \wedge \neg q) \vee r \equiv$$

$$\underset{\text{DeMorgan}}{\equiv} (\neg (p \vee q)) \vee r \underset{(*)}{\equiv} (p \vee q) \to r$$

2. (15 pts) Consider the relation R on the set of all integers $\mathbf{Z}$ defined by

$$(x, y) \in R \text{ if and only if } (x - y = 0 \text{ or } x - y = 1 \text{ or } x - y = -1) .$$

Answer "Yes" or "No" (2 pts) and briefly justify (1 pt). Correct answer with wrong justification receives only 1 pt.

(a) Is R reflexive?

Yes, $\forall x \in \mathbb{Z}$ $(x,x) \in R$ since $x - x = 0$

(b) Is R symmetric?

Yes, $\forall x, y \in \mathbb{Z}$ $(x,y) \in R \overset{def}{\iff} x - y \in \{0,1,-1\}$.
But $y - x = -(x-y)$, so if $x - y \in \{0,1,-1\} \Rightarrow y - x \in \{0,-1,1\}$. Thus $(x,y) \in R \Rightarrow (y,x) \in R$.

(c) Is R antisymmetric?

No. For example $(1,2) \in R$, $(2,1) \in R$ but $2 \neq 1$

(d) Is R transitive?

No. Example: $(1,2) \in R$, $(2,3) \in R$ (since $1-2 = 2-3 = -1$)
but $(1,3) \notin R$ (since $1-3 = -2 \notin \{0,1,-1\}$)

(e) Is R an equivalence relation on $\mathbf{Z}$?

No. R is reflexive and symmetric but it is not transitive

5. (8 pts) Let $P(m,n)$ the statement "m divides n", where the domain for both variables consists of all positive integers. (By "m divides n" we mean that $n = km$ for some integer k.) Determine the truth value of each of the following. No justification needed, just the answer will be graded.

(a) $P(5,23)$

False, $5 \nmid 23$.

(b) $\forall m \exists n\, P(m,n)$

True. Given any $m \in \mathbb{Z}_+$, let $n = 2m$. Obviously $m \mid 2m$.

(c) $\exists n \forall m\, P(m,n)$

False. There is no integer divisible by all positive integers.
If we assume there exist such an n take $m = n+1$. Obviously $n+1 \nmid n$.

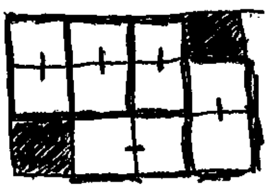
(d) $\exists m \forall n\, P(m,n)$

True. Let $m = 1$. Then $1 \mid n$ for any $n \in \mathbb{Z}_+$.

6. (14 pts) Decide whether is possible or not to cover each of the following with standard 2×1 dominoes. Justify your answer.

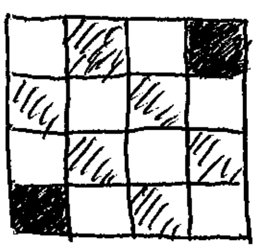
(a) (7 pts) A 3×4 checkerboard with two opposite corners removed.

Possible, here is one covering



We have not proved any general statement about when a certain region can be covered, thus you should describe or show a concrete covering.

(b) (7 pts) A 4×4 checkerboard with two opposite corners removed.



It is not possible. We color the 4×4 board with black and white in the standard way and remove two opposite corners. W.L.O.G. assume we removed 2 black corners! Then the region left has 8 white squares and 6 black squares. Such a region cannot be covered by dominoes since every domino would cover 1 black and 1 white square and this would imply that the #white squares equals #black squares

7. (10 pts) On the island of Smullyan there are two types of inhabitants, knights, who always tell the truth, and knaves, who always lie. You are on this island and meet two inhabitants A and B. A says "I am a knave or B is a knight" and B says nothing. Determine, if possible, what are A and B. Briefly describe your reasoning.

If A was a knave, the negation of its statement ~~[illegible]~~ must be true.
The negation of A's statement is I am not a knave and B is not a knight
It would follow that A is a knight but this contradicts the assumption that A is a knave.
The only possibility left is: A is a knight. In this case, his statement must be true. Since A's statement is an "or" statement, ~~[illegible]~~ his statement is true if B is a knight (and not true otherwise)
Thus ~~[illegible]~~ A, B must be both knights.

3. (8 pts) Write, in simple English, the negation of each of the following statements. Using logical connectors/quantifiers as an intermediate step may help. Do not start with a negation like "It is not true that ..."

(a) If John is enrolled at FIU then he has an FIU e-mail account.

$\neg(p \to q) \equiv p \wedge \neg q$

John is enrolled at FIU and he does not have an FIU e-mail account

(b) There is a problem in this exam that nobody in the class will solve correctly.

Original statement can be written as:

$\exists p \, \nexists x \; S(x,p)$, where $p \in$ set of problems on this exam, $x \in$ set of students in class, $S(x,p)$ = student x solves problem p.

Negation is

$\neg(\exists p \, \nexists x \, S(x,p)) \equiv \forall p \, \neg(\nexists x \, S(x,p)) \equiv \forall p \, \exists x \, S(x,p)$

Thus: "For every problem on the exam there exist (at least) a student in the class who will solve it correctly."

4. (16 pts) Suppose A and B are arbitrary sets.

(a) (8 pts) Show that $A - (A - B) = A \bigcap B$.

Sol. 1 Using $A \setminus B = A \cap \overline{B}$, we have

$$A \setminus (A \setminus B) = A \cap (\overline{A \cap \overline{B}}) \underset{\text{De Morgan}}{=} A \cap (\overline{A} \cup \overline{\overline{B}}) = A \cap (\overline{A} \cup B) =$$

$$\underset{\text{distributive law}}{=} (A \cap \overline{A}) \cup (A \cap B) = \emptyset \cup (A \cap B) = A \cap B$$

Sol. 2. The problem can be solved by showing the double inclusion element-wise

(b) (8 pts) Prove or disprove: If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ then $A \subseteq B$.

Sol. 1: We are given that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we want to show $A \subseteq B$.

Let $x \in A$. Then $\{x\} \subseteq A$, so $\{x\} \in \mathcal{P}(A)$. But $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ $\Rightarrow \{x\} \in \mathcal{P}(B)$

$\underset{\text{def of power set}}{\Rightarrow} \{x\} \subseteq B \Rightarrow \underline{x \in B}$. Thus $A \subseteq B$.

Sol. 2: As $A \subseteq A \underset{\text{def of power set}}{\Rightarrow} A \in \mathcal{P}(A) \xrightarrow{\mathcal{P}(A) \subseteq \mathcal{P}(B)} A \in \mathcal{P}(B) \Rightarrow$

$\underset{\text{def of power set}}{\Rightarrow} A \subseteq B$

8. (16 pts) Determine if each of the following statements is True or False. No justification needed, just the answer will be graded.

(a) The contrapositive of $p \to q$ is $\neg p \to \neg q$. False – contrapositive is $\neg q \to \neg p$

(b) For any set A, $\emptyset \subseteq A$. True

(c) For any set A, $\emptyset \in A$. False

(d) For any two sets A and B, $A \times B = B \times A$. False

(e) $\forall x(P(x) \lor Q(x)) \equiv (\forall x P(x)) \lor (\forall x Q(x))$ False

(f) The set $S = \{n^2 \mid n \in \mathbf{N}\}$ is countable. True

(g) The set of irrational numbers in the interval $(0, 1)$ is countable. False

(h) The function $f(x) = x^2 + 1$ is a bijection from $\mathbf{R}$ to $\mathbf{R}$.

False ; f is neither one-to-one, nor onto,

9. Choose ONE:

(a) (10 pts) Prove or disprove: The sum of a rational number with an irrational number is irrational.

(b) (14 pts) Show that the set of infinite bit strings is uncountable.

(a) ~~Statement~~ Statement is true. Proof by contradiction

Assume $\exists\, p \in \mathbb{Q}, q \notin \mathbb{Q}$ so that $p + q \in \mathbb{Q}$.

Since $p \in \mathbb{Q} \Rightarrow p = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, with $n \neq 0$.

Since $p + q \in \mathbb{Q} \Rightarrow p + q = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, with $b \neq 0$.

$$q = (p+q) - p = \frac{a}{b} - \frac{m}{n} = \frac{an - bm}{bn} \in \mathbb{Q}$$

since $an - bm \in \mathbb{Z}$ and $bn \in \mathbb{Z}$, $bn \neq 0$.

Contradiction $q \in \mathbb{Q}$ and $q \notin \mathbb{Q}$

Thus $\forall p \in \mathbb{Q}, \forall q \notin \mathbb{Q} \Rightarrow p + q \notin \mathbb{Q}$

(b) Cantor's diagonalization argument