Final Exam v.A - MAD 2014 - Summer A 2015 NAME: Solution Key

For full credit, when asked, you have to justify your answers. If "brief justification" is required, be brief, e.g "multiplicative rule" or "binomial theorem" will suffice (if correct). GOOD LUCK!

1. (15 pts)In each case, answer True or False. No justification is necessary for this problem – 3 pts each.

(a) For any sets A, B, $A \subseteq A \cup B$. True

(b) The set of infinite bit strings is countable. False

(c) The set $\{\sqrt{n} \mid n \in \mathbb{N}\}$ is countable. True

(d) For any undirected graph, the adjacency matrix A is symmetric, that is $A^T = A$. True

(e) $C(n,k) = C(n, n-k)$, for all $0 \leq k \leq n$. True

2. (10 pts) Define a set S recursively by $5 \in S$ and $9 \in S$ and if $x, y \in S$ then $x + y - 2 \in S$. Is $15 \in S$? Justify.

Yes, $15 \in S$.

$(5 \in S \wedge 5 \in S) \rightarrow 5 + 5 - 2 = 8 \in S$

$(9 \in S \wedge 8 \in S) \rightarrow 8 + 9 - 2 = 15 \in S$

3. (15 pts) Is it true that $(A - B) - C = (A - C) - B$ for all sets A, B, C? Answer and justify your answer. That is, prove the equality, or give a counter-example. A Venn diagram will help, but does not qualify as proof.

True. Proof (using ~~set identities~~ logic)

$x \in (A \setminus B) \setminus C \iff (x \in A \setminus B) \wedge x \notin C \iff$

$\iff (x \in A \wedge x \notin B) \wedge x \notin C \underset{\text{assoc. law}}{\iff} x \in A \wedge x \notin B \wedge x \notin C$

$\underset{\text{comm. law \& assoc. law}}{\iff} (x \in A \wedge x \notin C) \wedge x \notin B \iff (x \in A \setminus C) \wedge x \notin B \iff$

$\iff x \in (A \setminus C) \setminus B$

4. (15 pts) Give your answer and a brief explanation for each of the following. You don't have to simplify answers.

(a) A saleswoman has to visit eight different cities. She must begin her trip in a specified city (you can assume it is Miami) but she can visit the other seven cities in any order she likes. How many possible orders can the saleswoman use when visiting these cities?

$P(7,7) = 7!$ possible orders of the remaining 7 cities

(b) A club has 10 men and 15 women. In how many ways a committee of 2 men and 3 women can be selected? (The order of the people in the committee does not matter.)

$C(10,2) \cdot C(15,3)$ multiplicative property

(c) Another club has 8 women and 10 men. How many ways are there to select a committee of 4 people if the committee should contain at least one man and at least one woman? (The order of the people in the committee does not matter.)

$C(18,4) - C(8,4) - C(10,4)$ Complement rule

5. (15 pts) Use mathematical induction to prove ONE of these. Clearly indicate your choice.

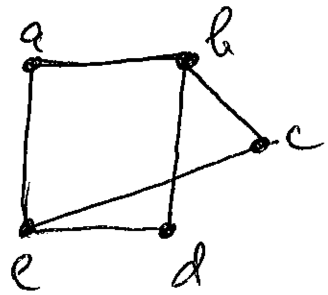
(A) Prove that 5 divides $n^5 - n$ for any non-negative integer n.

(B) $1^2+2^2+3^2+...+n^2 = \dfrac{n(n+1)(2n+1)}{6}$, for any $n \geq 1$.

Both were part of the suggested exercises, so I let you identify their solutions in the textbook

6. (30 pts) Let G be a simple graph with vertices $\{a, b, c, d, e\}$, with the adjacency matrix A given below.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$



(a) (5 pts) Draw G (or ask me to do it, but at a cost of up to 5 points).

For each of the remaining parts answer and briefly justify (5 pts each).

(b) Does G have an Euler circuit? An Euler path?

G does not have an Euler circuit, but it has an Euler path

Exactly two vertices of odd degree $\deg(b) = \deg(e) = 3$

$\deg(a) = \deg(c) = \deg(d) = 2$

(c) Is G connected?

Yes. Any two vertices are connected by a path. Note: ~~Prove~~ You could prove

(d) Is G bipartite ?

Yes. In fact G is isomorphic to $K_{2,3}$

(e) Is G isomorphic to W_4 (the Wheel with 5 vertices)?

No. W_4 contains a vertex of degree 4. G does not

(f) What is $\chi(G)$, the chromatic number of G?

$\chi(G) = 2$

7. (20 pts) In each case answer and very briefly justify – 4 pts each.

(a) How many edges does $K_{m,n}$ have?

$m \cdot n$

m vertices of deg n & n vertices of deg m.

$\sum \deg v = 2 \cdot |E| \qquad 2 \cdot |E| = 2m \cdot n \Rightarrow |E| = m \cdot n$

(b) What is the coefficient of y^4 in $(2y+3)^{10}$? You don't have to simplify the answer.

$C(10,6) \cdot 2^4 \cdot 3^6$ or $C(10,4) \cdot 2^4 \cdot 3^6$

(c) What is the exact value of $P(5,5)$?

$P(5,5) = 5! = 120$

What is the exact value of $C(10,3)$? (Writing the formula that leads to answer is enough justification.)

$C(10,3) = \dfrac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120$

(d) What is the value of the sum $C(n,0) - C(n,1) + C(n,2) - C(n,3) + \ldots + (-1)^n C(n,n)$?

0. $(1-1)^n$!!!

(e) How many numbers must be selected from the set $\{1,2,3,4,5,6,7,8\}$ to guarantee that at least one pair of these numbers add up to 9?

5. Pigeonhole principle with holes being the subsets $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, $\{4,5\}$,

8. Give an algebraic proof (10pts) or a combinatorial proof (15pts) for the identity

$$C(n,r)C(r,k) = C(n,k)C(n-k,r-k), \text{ where } 1 \leq k \leq r \leq n \ .$$

You may give two proofs, but you will get credit for only one of the two (the higher score).

see proof on p. B.

9. (15 pts) Give your answer and a brief explanation for each of the following. You don't have to simplify answers.

(a) How many bit strings of length 10 start with 00 or end with 111?

$2^8 + 2^7 - 2^5$ inclusion - exclusion

(b) How many bit strings of length 10 contain exactly four 0s?

$C(10,4)$ ← the number of ways to choose the places for the 0s.

(c) How many bit strings of length 10 contain exactly four 0s, with no two consecutive 0s? Hint: First place the 1s.

Box 1 Box 1 Box 1 Box 1 Box 1 Box 1 Box

you can choose boxes to contain one 0 in $C(7,4)$

So there are $C(7,4)$ bit strings satisfying the requirement

10. (15 pts) Consider the function $f : \mathbf{N} \to \mathbf{N}$ defined by $f(n) = \lfloor \frac{n}{3} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the number x (or "floor" of x).

(a) (4 pts) Find $f(10) =$ 3 $\qquad f(20) =$ 6

(b) (5 pts) Is the function f one-to-one? Justify your answer.

No $f(10) = f(11) = 3$

(c) (6 pts) Is the function f onto? Justify your answer.

Yes. Given $k \in \mathbb{N}$, $f(3k) = \lfloor \frac{3k}{3} \rfloor = k$

so f is onto