Name: Solution Key PanthID: ____________

Midterm Exam - v.1 MAD 2104 Summer A 2015

To receive credit you MUST SHOW ALL YOUR WORK.

1. (10 pts) Write, in simple English, the negation of each of the following statements. Using logical connectors/quantifiers as an intermediate step may help. Do not start with a negation like "It is not true that ...".

(a) Your warranty is good only if you bought your laptop less than 60 days ago.

p only if $q \equiv \neg q \to \neg p \equiv p \to q$ so

$\neg$(p only if q) $\equiv \neg(p \to q) \equiv p \wedge \neg q$

Your warranty is good and you bought the computer 60 days or more ago.

(b) Everyone in this room speaks French or Spanish.

$\forall x\,(F(x) \vee S(x))$ $F(x)$ = person x speaks French
 $S(x)$ = person x speaks Spanish

$\neg(\forall x\ F(x) \vee S(x)) \equiv \exists x\ \neg F(x) \wedge \neg S(x)$

* There is someone in this room who does not speak French and does not speak Spanish

2. (10 pts) Let $A_i = \{i, i+1, i+2, \ldots\}$, for every $i \in \mathbf{Z}^+$. Find the sets:

the largest set of the family (see the note)

(a) $\displaystyle\bigcup_{i=1}^{+\infty} A_i = A_1 = \{1, 2, 3, \ldots\} = \mathbb{Z}^+$

(b) $\displaystyle\bigcap_{i=1}^{5} A_i = A_5 = \{5, 6, 7, \ldots\}$

Note that: $A_1 = \{1,2,3,\ldots\}$, $A_2 = \{2,3,4,\ldots\}$, $A_3 = \{3,4,\ldots\}$, ...

so $A_{i+1} \subseteq A_i$ for all i.

3. (10 pts) Show that $(p \to q) \to r$ is not logically equivalent with $p \to (q \to r)$.

you could do a full truth table, but you could also notice that if p, q, r are all false, then

$(p \to q) \to r$ is false whereas $p \to (q \to r)$ is true.

Thus $(p \to q) \to r \not\equiv p \to (q \to r)$

* An answer like: "There is someone in this room who does not speak French and Spa
is logically misleading. You'd interpret it as saying that there is a person who
does not speak BOTH languages, whereas the ~~negation~~ correct interpretation of the
negation is that there is someone who does not speak either of the two languages.

4. (10 pts) Use logic (including DeMorgan's laws for logic) to prove this DeMorgan's laws for sets: $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

$x \in \overline{A \cap B} \Leftrightarrow x \notin A \cap B \Leftrightarrow \neg(x \in A \wedge x \in B) \overset{\text{Logic DeMorgan}}{\Leftrightarrow} \neg(x \in A) \vee \neg(x \in B) \Leftrightarrow$

$\Leftrightarrow x \notin A \vee x \notin B \Leftrightarrow x \in \overline{A} \vee x \in \overline{B} \Leftrightarrow x \in \overline{A} \cup \overline{B}$

Thus $\overline{A \cap B} = \overline{A} \cup \overline{B}$

5. (10 pts) Consider the relation $\mathcal{R}$ on the set of all real numbers $\mathbf{R}$ defined by $(x, y) \in R$ if and only if $x - y$ is an integer.

(8 pts) Show that $\mathcal{R}$ is an equivalence relation on the set real numbers $\mathbf{R}$. Briefly justify each property.

(2 pts) What is the equivalence class of π?

$\mathcal{R}$ is reflexive since $\forall x \in \mathbb{R}$ $x - x = 0 \wedge 0 \in \mathbb{Z}$, thus $\forall x \in \mathbb{R}$ $(x,x) \in \mathcal{R}$.

$\mathcal{R}$ is symmetric since: if $(x,y) \in \mathcal{R}$ ~~then~~ $x - y \in \mathbb{Z}$. But then $y - x = -(x-y)$ is also an integer then, so $(y,x) \in \mathcal{R}$.

$\mathcal{R}$ is transitive since: Let $(x,y) \in \mathcal{R}$ and $(y,z) \in \mathcal{R}$. Then $x - y = m \in \mathbb{Z}$ and $y - z = n \in \mathbb{Z}$

But then adding the two relations, get $x - z = m + n \in \mathbb{Z}$.

Thus if $(x,y) \in \mathcal{R} \wedge (y,z) \in \mathcal{R}$ then $(x,z) \in \mathcal{R}$

Equivalence class of π: $[\pi] = \{\pi + k \mid k \in \mathbb{Z}\} = \{\ldots, \pi - 2, \pi - 1, \pi, \pi + 1, \pi + 2, \ldots\}$

6. (10 pts) Let $A = \{a, b, c\}$. If any part of this problem is impossible, explain why.

(a) Give an example of a relation $\mathcal{R}$ on A which is neither symmetric nor anti-symmetric.

One of the possible examples

$\mathcal{R} = \{(a,b), (b,a), (a,c)\}$

(b) Give an example of a relation $\mathcal{S}$ on A which is reflexive, symmetric, but not transitive.

One of the possible examples

$S = \{(a,a), (b,b), (c,c), (a,b), (b,a), (b,c), (c,b)\}$

7. (16 pts) Determine if each of the following statements is True or False. No justification needed, just the answer will be graded.

F (a) A conditional statement is logically equivalent to its converse.

F (b) For any two sets A and B, if $A - B = \emptyset$ then $B - A = \emptyset$.

F (c) Any square $n \times n$ board can be tiled with dominoes (for all $n \geq 2$). when n is odd is not possible

T (d) If the domain for a, b is the set of positive real numbers, then $\forall a\ \exists b,\ ab = 2$. Given $a > 0$, take $b = \frac{2}{a} > 0$.

F (e) $\exists x(P(x) \wedge Q(x)) \equiv (\exists x P(x)) \wedge (\exists x Q(x))$

T (f) $\exists x(P(x) \vee Q(x)) \equiv (\exists x P(x)) \vee (\exists x Q(x))$

T (g) $\exists a, b \in \mathbf{N},\ |a^2 - b^3| = 1$ e.g. $a = 3$, $b = 2$ (also $a = 0$, $b = 1$ works, etc)

F (h) The function $f(x) = 2x + 1$ is a bijection from $\mathbf{Z}$ to $\mathbf{Z}$. It is not onto.

8. (10 pts) We would like to determine the relative salaries of three coworkers, Fred, Maggie, Janice, using the following facts. First, we know that there are no ties in their salaries. Then we know that if Fred is not the highest paid of the three, then Janice is. Finally, we know that if Janice is not the lowest paid, then Maggie is paid the most. Is it possible to determine the relative salaries of Fred, Maggie and Janice from what we know? If so, who is paid the most and who the least ? Explain your reasoning in words (for max credit, this should be a logical sequence of deductions, with clear reasons).

The order is: Fred highest, Maggie middle, Janice lowest

I gave about 3 pts for the correct answer and the rest of 7 pts for the correctness of your arguments.

One of many possible arguments:

Assume that Fred is not the highest paid. ~~Then~~ By second statement Janice is the highest paid. ~~This~~ 3rd statement now implies that Maggie is high paid and this is a contradiction with Janice being highest and the fact there are no ties in salaries.

Thus Fred must be the highest paid. This means that Maggie is not the highest paid so from contrapositive of second statement we get that Janice is the lowest paid.

9. Choose ONE (note the different values). If you choose (b) or (c), you may use without proof statement (a).

(a) (12 pts) Prove that $\sqrt{2}$ is not rational.

(b) (8 pts) Prove or disprove: There exist a rational number x and a rational number y so that x^y is irrational.

(c) (12 pts) Prove or disprove: There exist an irrational number x and an irrational number y so that x^y is rational.

For (a) see notes or text

(b) The statement is true: Let $x=2, y=\frac{1}{2}$. Both are rational but $x^y = 2^{\frac{1}{2}} = \sqrt{2}$ is irrational

(c) The statement is true (Done in class, see also Exp. 11 in section 1.8)

10. Choose ONE (note the different values):

(a) (8 pts) Prove that if $f: A \to B$ and $g: B \to C$ are both onto functions then the composition $g \circ f: A \to C$ is also onto.

(b) (12 pts) Prove that if $f: A \to B$ is an one-to-one function then for any subsets $S \subseteq A$, $T \subseteq A$,

$$f(S) \cap f(T) \subseteq f(S \cap T).$$

Sol. 1

(a) $f: A \to B$ onto $\Leftrightarrow f(A) = B$

$g: B \to C$ onto $\Leftrightarrow g(B) = C$

$(g \circ f)(A) = g(f(A)) = g(B) = C$ so $g \circ f$ is onto

Sol. 2: Let $c \in C$. Since $g: B \to C$ is onto $\exists\, b \in B$ s.t. $g(b) = c$.

Since $f: A \to B$ is onto, $\exists\, a \in A$ s.t. $f(a) = b$.

Thus, given $c \in C$, $\exists\, a \in A$ s.t. $g(f(a)) = g(b) = c$

so $g \circ f$ is onto.

(b) Let $y \in f(S) \cap f(T)$. Then $y \in f(S)$ and $y \in f(T)$, so

$\exists\, s \in S$ s.t. $f(s) = y$ and $\exists\, t \in T$ s.t. $f(t) = y$.

Since $f(s) = f(t) = y$ and f is one-to-one (by assumption) $\Rightarrow s = t$

Thus $s = t \in S \cap T \Rightarrow y \in f(S \cap T)$. We proved $f(S) \cap f(T) \subseteq f(S \cap T)$